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NOTE ON FRATTINI SUBGROUPS

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In this paper Frattini subgroups have been discussed. It has been proved that Next (C, A) is the Frattini subgroup of Ext (C, A). Frattini subgroups of cotorsion (algebraically compact) groups are cotorsion (algebraically compact). A number of results on groups of neat and pure-high extensions have been deduced.

The Frattini subgroup $\varphi(G)$ of G is the intersection of all maximal subgroups of G. The first Ulm subgroup G^1 of G is defined as the intersection of all the subgroups of G, that is

$$\varphi(G)=\bigcap_{p\in P} pG \textit{ and } G^1=\bigcap_{n\in N} nG.$$

For a positive integer n, the n^{th} Frattini subgroup of G is defined as the Frattini subgroup of $(n-1)^{th}$ Frattini subgroup of G. In this way a descending chain

$$G\geq\varphi(G)\geq\varphi(\varphi(G))\geq\cdots$$

of subgroups of G is obtained. Every Ulm subgroup of G is obviously contained in all the Frattini subgroups of G.

The exact sequence $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ is called neat exact if A is a neat subgroup of B. The elements of the group Next (C, A) are the neat exact sequences. The above exact sequence is a pure-high extension if and only if there exists a subgroup K of B such that A is maximal disjoint from K and $(A+K)/K$ is pure in B/K. The elements of the group $\text{Hext}_p(C, A)$ are the pure-high exact sequences.

In general we adopt the notations used in [1].

To start with we prove the following

LEMMA 1. *A group G is divisible if and only if $\varphi(G)=G$.*

PROOF. If G is divisible, then $nG=G$ for every $n\in N$ and

$$\varphi(G)=\bigcap_{p\in P} pG=G$$

Snce $\varphi(G)\subseteq pG$ for every $p\in P$ it follows that $G\subseteq PG$. Also $pG\subseteq G$ and thus G is divisible.

Next, we prove the following equivalence

THEOREM1. *For a group G, the following conditions are equivalent:*
(a) $\varphi(G)$ is divisible
(b) $\varphi(G)=G^1$
(c) $\varphi(G)$ is a neat subgroup of G.

PROOF. $a \Longleftrightarrow b$

Since for every $n \in N$, $n(\varphi(G)) \subseteq nG$. The divisibility of $\varphi(G)$ implies $\varphi(G) \subseteq nG$, that is $\varphi(G) \subseteq G^1$. The reverse inclusion is always true. Hence $\varphi(G)=G^1$.

Let $\varphi(G)=G^1$. It is clear that $G^1 \subseteq \varphi(\varphi(G))$ therefore, $\varphi(G) \subseteq \varphi(\varphi(G))$. The reverse inclusion holds. Hence by lemma 1, $\varphi(G)$ is divisible.

$a \Longleftrightarrow c$

If $\varphi(G)$ is divisible, then $\varphi(G)$ is a direct summand of G. This implies $\varphi(G)$ is pure and hence neat in G.

If $\varphi(G)$ is neat in G, then

$$p(\varphi(G))=\varphi(G) \cap pG, \text{ for every } P \in P.$$

But $\varphi(G) \subseteq pG$ and therefore $p(\varphi(G))=\varphi(G)$ implies $\varphi(G)$ is divisible.

Ulm subgroups of cotorsion groups are cotorsion and the Ulm factors of cotorsion groups are algebraically compact see theorem 54.3 of [1]. In the following theorem we prove that the Frattini subgroup of a Cotorsion group is again cotorsion and the factor group is algebraically compact.

THEOREM 2. *For a cotorsion group G the following hold*
(a) $\varphi(G)$ is cotorsion
(b) $G/\varphi(G)$ is algebraically compact.

PROOF. If G is a cotorsion group, then it is clear that $G/\varphi(G)$ is reduced and $\varphi(G/\varphi(G))=0$. The exact sequence

$$0 \longrightarrow \varphi(G) \longrightarrow G \longrightarrow G/\varphi(G) \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Hom}(Q, G/\varphi(G)) \longrightarrow \mathrm{Ext}(Q, \varphi(G)) \longrightarrow \mathrm{Ext}(Q, G).$$

The group of rationals Q, is divisible and $G/\varphi(G)$ is reduced it follows that the first group is O. Also the last group is O, and therefore $\varphi(G)$ is cotorsion.

The proof of the second part follows from Theorem 54.2 of [1] and from the fact that

$$(G/\varphi(G))^1 \subseteq \varphi(G/\varphi(G))=0$$

In general, the subgroups of algebraically compact groups are not algebraically compact. We prove the following.

THEOREM 3. *If G is algebraically compact, then $\varphi(G)$ is algebraically compact.*

PROOF. A group is algebraically compact if and only if its reduced part is. We may assume that $\varphi(G)$ is reduced. This implies G is reduced and by the remark following Corollary 38.2 of [1], $G^1=0$. But then $(\varphi(G))^1=0$, Theorem 54.2 of [1] completes the proof.

Pext(B, A) coincides with the first Ulm subgroup of Ext(B, A). We prove the following:

THEOREM 4. *For every group A and B*

$$Next(B, A)=\varphi(Ext(B, A))$$

PROOF. Theorem 53.3 of [1] states that the extension

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0$$

is divisible by a prime p if and only if A/pA is a direct summand of G/pA for all $p \in P$ (the set of primes). Since every direct summand is pure and hence neat it follows A/pA is neat in G/pA. Also pA is a neat subgroup of G and is contained in A. Adjust for simplicity $pA=\overline{A}$.

If $qg=a\in A$, $g\in G$, $q\in p$, then $q(g+\overline{A})=a+\overline{A}$ from which it follows that $a+\overline{A}=q(a_1+\overline{A})$, for some $a_1+\overline{A}\in A/\overline{A}$. This implies that $qa_1=a+\overline{a}$, where $\overline{a}\in \overline{A}$. Now $qg=a$ and $qa_1=a+\overline{a}$ imply that $q(a_1-g)=\overline{a}$. Neatness of $\overline{A}$ in G implies that there exists $\overline{\overline{a}}\in\overline{A}$ such that $q\overline{\overline{a}}=\overline{a}$. Thus

$$a=qa_1-\overline{a}=qa_1-q\overline{\overline{a}}=q(a_1-\overline{\overline{a}})$$

$(a_1-\overline{\overline{a}})\in A$ implies that A is neat in G and the extension is neat.

THEOREM5. *For a torsion group B,*

$$Hext_p(B, A)=\varphi(Pext(B, A))$$

PROOF. See Theorem 7 of [2]

We now record the results which can be deduced from the above theorems.

(a) Ext(B, A) is divisible if and only if Ext$(B, A)=$Next(B, A).

(b) For a torsion group B, Pext(B, A) is divisible if and only if Pext(B, A) $=$Hext$_p(B, A)$

(c) Ext(B, A) is divisible if B is torsion free.

(d) Next(B, A) is divisible if and only if Next$(B, A)=$Pext(B, A).

(e) For a torsion group B, Hext$_p(B, A)$ is divisible if and only if Hext$_p(B, A)$

$=\text{Pext}(B, A)^1$.

(f) $\text{Next}(B, A)$ is cotrosion.

(g) For a torsion roup B, $\text{Hext}_p(B, A)$ is a reduced cotrosion group.

(h) $\text{Next}(B, A)$ is algebraically compact, whenever $\text{Ext}(B, A)$ is.

(i) For a torsion group B, $\text{Hext}_p(B, A)$ is algebraically compact, whenever $\text{Pext}(B, A)$ is

(j) $\text{Ext}(B, A)/\text{Next}(B, A)$ is algebraically compact.

(k) For a torsion group B, $\text{Pext}(B, A)/\text{Hext}_p(B, A)$ is reduced algebraically compact.

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