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ON THE LARGE INDUCTIVE DIMENSION OF TYCHONOFF SPACE

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Introduction

We introduce a large inductive dimension function, f_x Ind X for a Tychonoff space X. We extend some previous results, that are known for normal space to arbitrary Tychonoff space. Moreover we show that f_x Ind X=Ind X for any normal space X. So that the theory of f Ind of a Tychonoff spaces may be consider as an extension of the theory of Ind of normal spaces.

In this paper all considered spaces are assumed to be Tychonoff. The family of all open (closed) subsets of a space X is denoted by $\tau(\tau^c)$. For $A\subseteq X$; O_A denotes the largest open subset of the Stone-cech compactification βX of X with the property that $O_A \cap X = A^0$. The closure of $A\subseteq X$ in βX will be denoted by bA, that is $bA=\overline{A}^{\beta X}$, and the boundary of A by **Fr** A.

1. Preliminaries

DEFINITION 1. (i) The subsets A_1 and A_2 of X are said to be completely separated in X, and we are write $A_1 f_x A_2$, if and only if there exists a continuous function $f: X \longrightarrow I$ such that $f(A_1)=\{0\}$ and $f(A_2)=\{1\}$.

(ii) A subset A of X is called f_X-neighbourhood of $B\subseteq X$, in symbol $A \underset{f}{\supset} B$, if and only if $B\ f_X (X|A)$.

(iii) We say that $Y\subseteq X$ has the property C^*, denoting this by $Y\in C^*(X)$, if and only if every continuous function $f: Y\longrightarrow I$ can be continuously extended over X.

Using that f_X is the finest proximity on a given space (X, τ) which is compatible with τ [2], [3] and [4], we deduce the following properties of f_X:

PROPOSITION 1. *Let A_1, A_2 and A are subsets of X, then:*

(i) $A_1 f_X A_2 \Longrightarrow \exists B_1,\ B_2 \subseteq X \ni B_i \underset{f}{\subset} A_i$ *and* $B_1 f_X B_2$.

(ii) $A_1 f_X A_2 \Longleftrightarrow \overline{A}_1 f_X \overline{A}_2 \Longleftrightarrow bA_1 \cap bA_2 = \phi$.

(iii) $A_1 f_X A_2 \Longrightarrow \overline{A}_2 \subseteq X | A_1$ *and* $A_2 \subseteq (X|A_1)^0$.

(*iv*) $(X|A_1)f_X(X|A_2) \Longrightarrow O_{A_1}\cup O_{A_2}=\beta X$.

(*v*) *If X is normal, then* $A_1 f_X A_2 \Longleftrightarrow \overline{A}_1\cap\overline{A}_2=\phi$.

(*vi*) $A_1 \underset{f}{\supset} A_2 \Longrightarrow A_1\supseteq A_2$.

(*vii*) $A_1\supseteq A_2 \underset{f}{\supset} A_3 \supseteq A_4 \Longrightarrow A_1 \underset{f}{\supset} A_4$.

(*viii*) $A_1 \underset{f}{\supset} A_2 \Longrightarrow X|A_2 \underset{f}{\supset} X|A_1$.

(*ix*) $A_1 \underset{f}{\supseteq} A_2 \Longrightarrow A_1 \underset{f}{\supseteq} \overline{A}_2$ *and* $A_1^0 \underset{f}{\supset} A_2$.

(*x*) $A_i \underset{f}{\supset} B_i$ *for* $i=1, 2, \cdots, n \Longrightarrow \bigcap_{i=1}^n A_i \underset{f}{\supset} \bigcap_{i=1}^n B_i$ *and* $\bigcup_{i=1}^n A_i \underset{f}{\supset} \bigcup_{i=1}^n B_i$.

(*xi*) $A_1 \underset{f}{\supset} A_2 \Longrightarrow O_{A_1}\supseteq bA_2$.

(*xii*) $bA=\beta X|O_{(X|A)}$.

(*xiii*) $O_A=O_{A^0}$.

(xiv) $bO_A=bA\ \forall\ A\in\tau$.

(*xv*) $O_{A_1\cap A_2}=O_{A_1}\cap O_{A_2}$.

(*xvi*) $O_{\bigcup_\lambda A_\lambda}\supseteq\bigcup_\lambda O_{A_\lambda}$.

(*xvii*) *If* $A_1\cap A_2=\phi$ *and* $A_1, A_2\in\tau$, *then* $O_{A_1}\cup O_{A2}=O_{A1\cup A2}$.

(*xviii*) $\mathbf{Fr}\ O_A=b\,\mathbf{Fr}\ A\ \forall\ A\in\tau$.

(*xix*) *If* $Y\in C^*(X)$ *and* $A_1, A_2\subseteq Y$, *then* $A_1 f_X A_2 \Longleftrightarrow A_1 f_Y A_2$.

(*xx*) *If* $Y\in C^*(X)$, *then* $bx=\beta Y$.

DEFINITION 2. Let $Y\subseteq X$. The triple (L, V_1, V_2), where V_1 and V_2 are disjoint open subsets of Y, is called f_X-partition between $A_1, A_2\subseteq Y$ in Y if and only if $Y|L=V_1\cup V_2$ and $A_i\ f_X(Y|V_i)$ for $i=1, 2$. The following lemma is obvious.

LEMMA 1. *Let* $Y\subseteq X$ *and* $F_1, F_2\subseteq Y$ *such that* $F_1 f_X F_2$. *If* (L, V_1, V_2) *is a partition between* bF_1 *and* bF_2 *in* βX *(in sense* |1|), *then* $(L\cap Y, V_1\cap Y, V_2\cap Y)$ *is a* f_X*-partition between* F_1 *and* F_2 *in* Y.

LEMMA 2. *Let* $F_1, F_2\subseteq X$ *and* $F_1 f_X F_2$. *If* (L, V_1, V_2) *is a* f_X*-partition between* F_1 *and* F_2 *in* X, *then* (bL, O_{V_1}, O_{V_2}) *is a partition between* bF_1 *and* bF_2 *in* βX.

PROOF. From definition 2 we have;

$$X|L=V_1\cup V_2 \text{ and } V_1 \underset{f}{\supseteq} F_i \text{ for } i=1, 2,$$

Let $L_i=L\cup V_i$, then it is clear that

$$L_i\in\tau^C,\ F_1 f_X F_2 \text{ and } L_2 f_X L_1.$$

Hence $bL_1 \cap bF_2 = bL_2 \cap bF_1 = \phi$, by proposition (1-ii). Using proposition (1-xii) we have

$$bF_1 \subseteq \beta X \mid bL_2 = O_{X \mid L_1} = O_{V_1}$$

and $bF_2 \subseteq \beta X \mid bL_1 = O_{X \mid L_2} = O_{V_2}$.

Since $V_1 \cap V_2 = \phi$, proposition (1-xv) imply ·

$$O_{V_1} \cap O_{V_2} = O_{V_1 \cap V_2} = \phi.$$

Now $X \mid L = V_1 \cup V_2$, by proposition (1-xvii) we have;

$$O_{X \mid L} = \beta X \mid bL = O_{V_1 \cup V_2} = O_{V_1} \cup O_{V_2}.$$

Thus the triple (bL, O_{V_1}, O_{V_2}) is a partition between bF_1 and bF_2 in βX.

2. The large inductive of Tychonoff space

DEFINITION 3. Let (X, τ) be a Tychonoff space and $Y \subseteq X$. The f_X-large inductive dimension of Y, denoted by f_X Ind Y, is defined inductively as follows:

f_X Ind $Y = -1$ iff $Y = \phi$. For a non-negative integer n, f_X Ind $Y \leq n$ means that for each pair of subsets F_1 and F_2 of Y, for which $F_1 f_X F_2$, there exists a f_X-partition (L, V_1, V_2) between F_1 and F_2 in Y such that

f_X Ind $L \leq n-1$,

f_X Ind $Y = n$ iff $n-1 < f_X$ Ind $Y \leq n$ and

f_X Ind $Y = \infty$ iff there is no n for which f_X Ind $Y \leq n$.

Using proposition (i-v, xx) and the above definiticn, one may easily prove the following three theorems:

THEOREM 1: *If $Y \in C^*(X)$, then f_XInd $Y = f_Y$ Ind Y.*

THEOREM 2. *If $Z \subseteq Y \subseteq X$ and $Y \in C^*(X)$, then f_X Ind $Z = f_Y$ Ind Z.*

THEOREM 3. *If X is a normal space, then f_X Ind $X =$ Ind X.*

THEOREM 4. *If $Z \subseteq Y \subseteq X$, then f_X Ind $Z \leq f_X$ Ind Y.*

PROOF. Let f_X Ind $Y = k$. For $k = -1$ the result is trivial. We assume its validity for $k < n$ and suppose $k = n$.

Let F_1, $F_2 \subseteq Z$ be suth that $F_1 f_X F_2$. Then there exists a f_X-partition (L, U_1, U_2) between F_1 and F_2 in Y, for which f_X Ind $L \leq n-1$. Evidently the triple $(L \cap Z, U_1 \cap Z, U_2 \cap Z)$ is a f_X-partition between F_1 and F_2 in Z, and hence by inductive assumption,

$$f_X \text{ Ind } (L\cap Z \leq f_X \text{ Ind } L \leq n-1.$$

Thus
$$f_X\text{Ind } Z \leq f_X \text{ Ind } Y.$$

COROLLARY 1. *If $Y\subseteq X$, then f_X Ind $Y\leq f_X$* Ind *X*.

COROLLARY 2. *If $Y\in C^{*}(X)$, then f_Y Ind $Y\leq f_X$ Ind X.*

THEOREM 5. *If $Y\subseteq X$,* then *f_X Ind $Y\leq$ Ind bY.*

PROOF. Let Ind $by=k$, for $k=-1$ the result is trivial. We assume its validity for $k\leq n-1$ and suppose that $k=n$.

Let F_1 and F_2 be (closed) subsets of Y such that $F_1 f_X F_2$. Then bF_1 and bF_2 are disjoint closed subsets of bY. Thus there exists a partition (L, V_1, V_2) between bF_1 and bF_2 in bY such that Ind $L\leq n-1$.

From lemma 1 the triple $(L\cap Y, V_1\cap Y, V_2\cap Y)$ is a f_X-partition between F_1 and F_2 in Y. Since $b(L\cap Y)\subseteq L$, then by theoren 2.2.1 in [1],

Ind $b(L\cap Y)\leq$ Ind $L\leq n-1$, and hence by inductive assumption
$$f_X \text{ Ind } (L\cap Y)\leq n-1.$$

Thus
$$f_X \text{ Ind } Y\leq n.$$

COROLLARY. *f_X Ind $X\leq$ Ind βX, for every Tychonoff shace X.*

THEOREM 6. *Ind $Y\leq f_X$ Ind Y for $Y\subseteq X$.*

PROOF. It is easy to prove it by applying the induction with respect to f_X Ind Y, noting that
$$\forall F\in\tau^C,\ X\notin F \text{ iff } \{x\}\ f_X F.$$

From theorems 5 and 6 we have

THEOREM 7. *ind $X\leq f_X$ Ind $X\leq$ Ind βX for every Tychonoff space X.*

THEOREM 8. *If f_X Ind $X=0$, then Ind $\beta X=0$.*

PROOF. Let F_1 and F_2 are disjoint closed subsets of βX. Then $F_1\cap X$ and $F_2\cap X$ are completely separated in X. Since f_X Ind $X=0$, there exists a f_X-partition (L, V_1, V_2) between $F_1\cap X$ and $F_2\cap X$ in X such that $L=\phi$. By lemma 2, (bL, O_{V_1}, O_{V_1}) is a partition in βX between F_1 and F_2. Thus Ind $\beta X=0$.

From theorems 8 and 5 we have:

COROLLARY. 1. *f_X Ind $X=0$ if and only if Ind $\beta X=0$.*

From the above corollary, theorem 1_t in [5] and theorem 1.6.11 in [1] we have,

COROLLARY 2. *f_X Ind $X=0$ iff dim $X=0$.*

THEOREM 9. *f_X Ind $X\leq n$ iff for every (closed) subset F of X and each open subset U of X such that $U\underset{f}{\supseteq}F$, there exists an open subset U^* of X such that*

$$U\underset{f}{\supseteq}U^{*}\underset{f}{\supseteq}F \text{ and } f_X \text{ Ind } \mathbf{Fr}\ U^{*}\leq n-1.$$

PROOF. Let f_X Ind $X\leq n$. Consider a (closed) subset F of X and an open subset U of X with $U\underset{f_x}{\supset}F$.

Since $Ff_X(X|U)$, then there exists a f_X-partition (L, V_1, V_2) in X between F and $X|U$ satisfying f_X Ind $L\leq n-1$.

Hence $$X|L=V_1\cup V_2,\ V_1\cap V_2=\phi,\ V_1\underset{f_x}{\supset}F$$
and $$V_2\underset{f_x}{\supseteq}X|U.$$

Thus by proposition (1-viii), $U\underset{f_x}{\supseteq}X|V_2\supseteq V_1\underset{f_x}{\supseteq}F$. Since $\mathbf{Fr}\ V_1\subseteq(X|V_1)\cap(X|V_2)$ $=X|V_1\cup V_2=L$, then by theorem 4, f_X Ind $\mathbf{Fr}\ V_1\leq f_X$ Ind $L\leq n-1$. So the set V_1 is the required one.

Conversely, let (X,τ) be a Tychonoff space satisfying the conditions of the theorem, consider F_1, $F_2\subset X$ such that $F_1f_XF_2$.

By the definition of $\underset{f_x}{\supseteq}$ we have $X|F_2\underset{f_x}{\supseteq}F_1$.

From the given condition, there exists $U\varepsilon\tau$ such that $X|F_2\underset{f_x}{\supseteq}U\underset{f_x}{\supseteq}F_1$ and f_X Ind $\mathbf{Fr}\ U\leq n-1$.

Using proposition 1 it is easy to see that the triple (Fr U, U, $X|\overline{U}$) is a f_X-partition between F_1 and F_2 in X. So that f_X Ind $X\leq n$.

COROLLARY. *Let f_X Ind $X=n$, then for every $k=0, 1, 2, \cdots, n-1$, The space X contains a closed subspace Y_k such that f_X Ind $Y_k=K$.*

PROOF. The proof of this corollary is similar to the proof of the theorem 1.5.1 in [1].

DEFINITION 4. A f_X-base for a Tychonoff space (X, τ) is a subfamily β of 2^X such that

$A_1 f_X A_2$ implies the existence of $V_1,\ V_2 \in \beta$ such that $V_i \supseteq A_i$ for $i=1,\ 2$ and $V_1 f_X V_2$.

LEMMA 3. *A sub-family $\beta \subseteq 2^X$ is a f_X-base for a Tychonoff space $(X,\ \tau)$ if and only if*

$$\forall F \subseteq X,\ \forall V \underset{f_x}{\supseteq} F\ \exists L \in \beta \in V \underset{f_x}{\supseteq} L \underset{f_x}{\supseteq} F.$$

PROOF. Let β be a f_X-base for (X, τ) and $F,\ V \subseteq X$ such that $V \underset{f_x}{\supseteq} F$. Since $F f_X (X|V)$, then by proposition $(1-i)$, there exist $U_1,\ U_2 \subseteq X$ such that

$$U_1 \underset{f_x}{\supseteq} F,\ U_2 \underset{f_x}{\supseteq} X|V \text{ and } U_1 f_X U_2.$$

Since β is a f_X-base, there exist $L,\ L^* \varepsilon \beta$ such that

$$L \supseteq U_1,\ L^* \supseteq U_2 \text{ and } L\ f_X\ L^*.$$

and hence
$$L \supseteq C \underset{f_x}{\supseteq} F,\ L^* \supseteq U_2 \underset{f_x}{\supseteq} X|V$$

and
$$X|L^* \underset{f_x}{\supseteq} L.$$

From proposition (1-vii, viii) it follows that

$$L \underset{f_x}{\supseteq} F,\ V \underset{f_x}{\supseteq} X|L^* \text{ and } X|L^* \underset{f_x}{\supseteq} L.$$

Thus

$$V \underset{f_x}{\supseteq} L \underset{f_x}{\supseteq} F.$$

Conversely, let β be a sub-family of 2^X such that $V \underset{f_x}{\supseteq} L \underset{f_x}{\supseteq} F$ whenever $V \underset{f_x}{\supseteq} F$. Assuming that $F_1 f_X F_2$, we have $X|F_2 \underset{f_x}{\supseteq} F_1$. Thus, there is $L \in \beta$ such that $X|F_2 \underset{f_x}{\supseteq} L \underset{f_x}{\supseteq} F_1$. Since $X|F_2 \underset{f_x}{\supseteq} L$, then $X|L \underset{f_x}{\supseteq} F_2$ and there is $L^* \in \beta$ such that $X|L \underset{f_x}{\supseteq} L^* \underset{f_x}{\supseteq} F_2$.

It is clear that $L f_X L^*$.

Thus β is a f_X-base for $(X,\ \tau)$.

From the above lemma and theorem 8 one can easily prove the following:

THEOREM 10. *A space $(X,\ \tau)$ has f_X Ind $X \leq n$ iff it has a f_X-base β consisting of open sets such f_X Ind Fr $L \leq n-1$ for every $L \in \beta$.*

LEMMA 4. *Let $(X,\ \tau)$ be space and $Y \in \tau$. If f_X Ind $Y \leq n$ and $F_1,\ F_2 \subseteq X$ such that $F_1 f_X F_2$, then there exists a f_X-partition $(L,\ V_1,\ V_2)$ between F_1 and F_2 in X such that f_X Ind $L \cap Y \leq n-1$.*

PROOF. Since $F_1 f_X F_2$, then by proposition (1-i, ix) there exist $U_1,\ U_2 \in \tau$ such that $U_i \underset{f_X}{\supseteq} F_i$ and $U_1 f_X U_2$. Hence $U_1 \cap Y f_X \overline{U}_2 \cap Y$. Thus there exists a f_X-partition (L^*, U_1^*, U_2^*) between $\overline{U}_1 \cap Y$ and $\overline{U}_2 \cap Y$ in Y such that f_X Ind $L^* \leq n-1$.

Consider $V_i = U_i \cup U_i^*$ and $L = X | (V_1 \cup V_2^*)$, then the triple (L, V_1, V_2) is f_X-partition between F_1 and F_2 in X satisfying the condition $L \cap Y = L^*$.

Hence f_X Ind $L \cap Y \leq n-1$.

THEOREM 11. *If $X = A \cup B$ and A is an open in X, then*

$$f_X \; Ind \; X \leq f_X \; Ind \; A + f_X \; Ind \; B + 1.$$

PROOF. Taking lemma 4 into consideration the proof of this theorem is similar to the proof of theorem 2.2.5 in [1].

COROLLARY. *If $X = Y \cup Z$ such that Y is closed, then*

$$f_X \; Ind \; X \leq f_X \; Ind \; Y + f_X \; Ind \; Z + 1.$$

PROOF. Since $X = Y \cup Z = Y \cup (X|Y)$ and $(X|Y \subseteq Z$, theorem 11 implies.

$$f_X \text{ Ind } X = f_X \text{ Ind } Y \cup (X|Y) \leq f_X \text{ Ind } Y + f_X \text{ Ind } (X|Y) + 1$$

From theorem 4 we have f_X Ind $(X|Y) \leq f_X$ Ind Z.

Hence $$f_X \text{ Ind } (Y \cup Z) \leq f_X \text{ Ind } Y + f_X \text{ Ind } Z + 1.$$

COROLLARY 2. *If $X = Y \cup Z$ and $Y, Z \in C^*(X)$, then*

$$f_X \; Ind \; (Y \cup Z) \leq f_Y \; Ind \; Y + f_Z \; Ind \; Z + 1.$$

COROLLARY 3. *If $X = \bigcup_{i=1}^{n} Y_i$, $Y_i \in C^*(X)$ for $i = 1, 2, \cdots, n$ and f_{Y_i} Ind $Y_i \leq 0$, then f_X Ind $X \leq n$.*

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