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NONLINEAR MAPPINGS IN METRIC AND HAUSDORFF SPACES AND THEIR COMMON FIXED POINT

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Abstract: In the first section of this paper two common fixed point results for four nonlinear mappings which are pairwise commuting and only two of them being continuous have been given on a complete metric space and on a compact metric space respectively which generalize the results of Mukherjee [2] and Yeh [4]. Further two common fixed point theorems have been established for two finite families of nonlinear mappings, with only one family being continuous. In another section we extend Theorem 3 and Theorem 4 of Mukherjee [2] for common fixed point of four continuous mappings on a Hausdorff space and on a compact metric space respectively. In the same spaces, these two results have been further generalized for two finite families of continuous mappings.

1. Introduction

Mukherjee [2] proved following common fixed point theorem for a pair of commuting nonlinear mappings.

THEOREM A. *Let f and g be mappings of a complete metric space into itself with f continuous. Let f and g commute with each other and $g(X)\subset f(X)$. Also let g sasitfy the following condition*

$$d(g(x),\ g(y))\leq a_1d(g(x),\ f(x))+a_2d(g(y),\ f(y))+a_3d(g(x),\ f(y))$$
$$+a_4d(g(y),\ f(x))+a_5d(f(x),\ f(y))$$

with $a_i\geq 0$ for all i and $a_1+a_2+a_3+2a_4+a_5<1$, then f and g have a unique common fixed point in X.

And Yeh [4] proved a common fixed point result for three continuous mappings which goes as follows.

THEOREM B. *Let E, F and T are three continuous self mappings of a com-*

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plete metic space (X, d) satisfying the following conditions:

(1) $ET=TE$, $FT=TF$, $E(X)\subset T(X)$ *and* $F(X)\subset T(X)$,

(2) $$d(Ex, Fy)\leq a(d(Tx, Ty))d(Tx, Ty) + b(d(Tx, Ty))[d(Tx, Ex)+d(Ty, Fy)] + c(d(Tx, Ty))[d(Tx, Fy)+d(Ty, Ex)].$$

for all $x, y\in X$, *where* a, b *and* c *are monotonically decreasing functions from* R^+ *into* $[0, 1]$ *satisfying* $a(t)+2b(t)+2c(t)<1$ *for all* $t\in R^+$, *then* E, F *and* T *have a unique common fixed point.*

In what follows we give our first common fixed point result for four mappings under some generalized conditions than that of Mukherjee [2] and Yeh [4].

THEOREM 1. *Let* f_i *and* g_i, $(i=1,2)$ *be four nonlinear self mappings of a complete metric space* (X, d) *with each* f_i *being continuous and let* f_i *and* g_i *satisfy the following conditions.*

(3) $f_1g_2=g_2f_1$, $f_2g_1=g_1f_2$ *and* $g_i(X)\subset f_i(X)$ *for* $i=1$ *and* 2.

(4) $$d(g_1(x), g_2(y))\leq a_1d(g_1(x), f_2(x))+a_2d(g_2(y), f_1(y)) + a_3d(g_1(x), f_1(y))+a_4d(g_2(y), f_2(x)) + a_5d(f_2(x), f_1(y))$$

for all $x, y\in X$, *where* $a_j\leq 0$ *for* $j=1, \cdots, 5$ *and* $\sum_{j=1}^{5} a_j+a_3<1$. *Then all* f_i *and* g_i *have a unique common fixed point in* X.

PROOF. Let for any arbitrary $x_0\in X$ $x_1\in X$ be such that $g_1(x_0)=f_1(x_1)=y_1$ (say). For this x_1 let $x_2\in X$ be such that $g_2(x_1)=f_2(x_2)=y_2$(say) and so on. Therefore in general we have

$$g_1(x_{2n})=f_1(x_{2n+1})=y_{2n+1}\text{(say) and}$$
$$g_2(x_{2n+1})=f_2(x_{2n+2})=y_{2n+2}\text{(say)},\ n=0, 1, \cdots.$$

Now from (4),

$$\begin{aligned} d(y_3, y_2)&=d(g_1(x_2), g_2(x_1)) \\ &\leq a_1d(g_1(x_2), f_2(x_2))+a_2d(g_2(x_1), f_1(x_1)) \\ &\quad + a_3d(g_1(x_2), f_1(x_1))+a_4d(g_2(x_1), f_2(x_2)) \\ &\quad + a_5d(f_2(x_2), f_1(x_1)) \end{aligned}$$

which on simplification gives,

$$d(y_3, y_2)\leq \frac{a_2+a_3+a_5}{1-a_1-a_3d}(Y_2, y_1)$$

$$\leq \alpha d(y_2, y_1), \text{ where } \alpha = \frac{a_2+a_3+a_5}{1-a_1-a_3} < 1.$$

Similarly in general, we get

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n}) \text{ and}$$
$$d(y_{2n+1}, Y_{2n+2}) \leq \alpha d(y_{2n}, y_{2n+1}), \ n=1, 2, \cdots.$$

Hence the sequence $\{y_n\}$ is Cauchy, since $\alpha<1$, let $\{y_n\}$ converges to some to some $t \in X$ due to completeness of X. Therefore

$$y_{2n+1} = g_1(x_{2n}) = f_1(x_{2n+1}) \longrightarrow t \text{ and}$$
$$y_{2n+2} = g_2(x_{2n+1}) = f_2(x_{2n+2}) \longrightarrow t.$$

Not since f_1 and f_2 are continuous, we have by (3) that

$$g_1(f_2(x_{2n})) = f_2(g_1(x_{2n})) = f_2(y_{2n+1}) \longrightarrow f_2(t) \text{ and}$$
$$g_2(f_1(x_{2n+1})) = f_1(g_2(x_{2n+1})) = f_1(y_{2n+2}) \longrightarrow f_1(t).$$

Further denoting $2n$ by m, we have

$$\begin{aligned} &d(g_1(f_2(x_m)), g_2(f_1(x_{m+1}))) \\ &\leq a_1 d(g_1(f_2(x_m)), f_2(f_2(x_m))) \\ &+a_2 d(g_2(f_1(x_{m+1})), f_1(f_1(x_{m+1}))) \\ &+a_3 d(g_1(f_2(x_m)), f_1(f_1(x_{m+1}))) \\ &+a_4 d(g_2(f_1(x_{m+1})), f_2(f_2(x_m))) \\ &+a_5 d(f_2(f_2(x_m)), f_1(f_1(x_{m+1}))) \end{aligned}$$

On taking limits of both the sides of the above after a little simplification, we get

$$\begin{aligned} d(f_2(t), f_1(t)) \leq\ & a_1 d(f_2(t), f_2(t)) + a_2 d(f_1(t), f_1(t)) \\ &+a_3 d(f_2(t), f_1(t)) + a_4 d(f_1(t), f_2(t)) \\ &+a_5 d(f_2(t), f_1(t)) \end{aligned}$$

or, $(1-a_3-a_4-a_5) d(f_1(t), f_2(t)) \leq 0$

but $1-a_3-a_4-a_5>0$ and therefore

$$d(f_1(t), f_2(t)) = 0, \text{ i.e., } f_1(t) = f_2(t).$$

Now,
$$\begin{aligned} d(g_1(f_2(x_m)), g_2(t)) \leq\ & a_1 d(g_1(f_2(x_m)), f_2(f_2(x_m))) \\ &+a_2 d(g_2(t), f_1(t)) + a_3 d(g_1(f_2(x_m)), f_1(t)) \\ &+a_4 d(g_2(t), f_2(f_2(x_m))) + a_5 d(f_2(f_2(x_m)), f_1(t)) \end{aligned}$$

This on using (3) viz. $g_1 f_2 = f_2 g_1$ and then on taking limits of both the sides gives

$$\begin{aligned} d(f_2(t), g_2(t)) \leq\ & a_1 d(f_2(t), f_2(t)) + a_2 d(g_2(t), f_1(t)) \\ &+a_3 d(f_2(t), f_1(t)) + a_4 d(g_2(t), f_2(t)) \\ &+a_5 d(f_2(t), f_1(t)) \end{aligned}$$

or, $(1-a_2-a_4)d(f_2(t),\ g_2(t))\leq 0$ since $1-a_2-a_4>0$, we have $d(f_2(t),\ g_2(t))=0$ or, $g_2(t)=f_2(t)=f_1(t)$. Further

$$\begin{aligned}d(g_1(t),\ g_2(f_1(x_{m+1}))\leq & a_1d(g_1(t),\ f_2(t))\\ &+a_2d(g_2(f_1(x_{m+1})),\ f_1(f_1(x_{m+1})))+a_3d(g_1(t),\ f_1(f_1(x_{m+1})))\\ &+a_4d(g_2(f_1(x_{m+1})),\ f_2(t))+a_5d(f_2(t),\ f_1(f_1(x_{m+1})))\end{aligned}$$

Using $g_2f_1=f_1g_2$ in the above we have in the limiting case

$$\begin{aligned}d(g_1(t),\ f_1(t))\leq & a_1d(g_1(t),\ f_2(t))+a_2d(f_1(t),\ f_1(t))\\ &+a_3d(g_1(t),\ f_1(t))+a_4d(f_1(t),\ f_2(t))\\ &+a_5d(f_2(t),\ f_1(t))\end{aligned}$$

or, $(1-a_1-a_3)d(g_1(t),\ f_1(t))\leq 0$ but $1-a_1-a_3<0$, therefore we have

$$d(g_1(t),\ f_1(t))=0,\ \text{i.e.,}\ g_1(t)=f_1(t).$$

Thus we have
$$g_1(t)=f_1(t)=f_2(t)=g_2(t) \tag{5}$$

Next we show that $g_i(t)=f_i(t)$, i, $j=1,\ 2$ is the unique common fixed point of f_i and g_i.

By (3), $f_2(g_1(t))=g_1(f_2(t))=g_1(g_1(t))$, therefore

$$\begin{aligned}d(g_1(g_1(t)),\ g_2(t))\leq & a_1d(g_1(g_1(t)),\ f_2(g_1(t)))+a_2d(g_2(t),\ f_1(t))\\ &+a_3d(d_1(g_1(t)),\ f_1(t))+a_4d(g_2(t),\ f_2(g_1(t)))\\ &+a_5d(f_2(g_1(t)),\ f_1(t))\end{aligned}$$

or, $(1-a_3-a_4-a_5)d(g_1(g_1(t)),\ g_1(t))\leq 0$ as $1-a_3-a_4-a_5>0$, we have

$$d(g_1(g_1(t)),\ g_1(t))=0,\ \text{i.e.,}\ g_1(g_1(t))=g_1(t).$$

Further since

$$f_1(g_1(t))=f_1(g_2(t))=g_2(f_1(t))=g_2(g_1(t))$$

then putting $x=t$ and $y=g_1(t)$ in (4), we can similarly show that $g_2(g_1(t))=g_1(t)$. Thus $f_1(g_1(t))=f_1(g_2(t))=g_2(f_1(t))=g_2(g_1(t))=g_1(t)$ and $f_2(g_1(t))=g_1(f_2(t))=g_1(g_1(t))=g_1(t)$. Hence $g_1(t)$ is a common fixed point of g_1, g_2, f_1 and f_2. Therefore by (5) we claim that $g_2(t)=f_2(t)=f_1(t)=g_1(t)$ is a common fixed point of g_1, g_2, f_1 and f_2. To show uniqueness let $g_1(s)=g_2(s)=f_1(s)=f_2(s)$ be another common fixed point of g_1, g_2, f_1 and f_2, then

$$\begin{aligned}d(g_1(t),\ g_2(s))\leq & a_1d(g_1(t),\ f_2(t))+a_2d(g_2(s),\ f_1(s))\\ &+a_3d(g_1(t),\ f_1(s))+a_4d(g_2(s),\ f_2(t))\\ &+a_5d(f_2(t),\ f_1(s))\end{aligned}$$

or, $(1-a_3-a_4-a_5)d(g_1(t),\ g_2(s))\leq 0$, since $1-a_3-a_4-a_5>0$, we have the required result.

Further if we take $(x,\ d)$ *to* be a compact metric space in the above theorem, we have the following result, which we state without proof.

THEOREM 2. *Let f_i and g_i, $i=1, 2$ be four nonlinear self mappings of a compact metric space (X, d) with each f_i being continuous. Let f_i and g_i satisfy the following conditions*

(6) $f_1g_2=g_2f_1$, $f_2g_1=g_1f_2$ *and* $g_i(X)\subset f_i(X)$ *for* $i=1$ *and* 2

(7) $$d(g_1(x), g_2(y))\leq a_1d(g_1(x), f_2(x))+a_2d(g_2(y)), f_1(y)) +a_3d(g_2(x), f_1(y))+a_4d(g_2(y), f_2(x)) +a_5d(f_2(x), f_1(y))$$

for all $x, y\in X$, where $a_j\geq 0$, $j=1, \cdots, 5$ with $\sum_{j=1}^{5}a_j+a_3=1$. Then all f_i and g_i have a unique common fixed point in X.

REMARK 1. Taking $f_1=f_2$ and $g_1=g_2$, we get Theorem 1 and Theorem 2 of Mukherjee [2] as corollaries of our Theorem 1 and Theorem 2 respectively.

REMARK 2. Taking $g_1=E$, $g_2=F$, $f_1=f_2=T$ and $a_5=a$, $a_1=a_2=b$ and $a_3=a_4=c$ in our Theorem 1, we have a generalized result of Yeh [4] with a simplification that a, b and c are real constants there. If we further take g_1 and g_2 continuous, then the proof becomes much easier.

THEOREM 3. *Let $\{f_i\}$ and $\{g_i\}$, $i=1, 2, \cdots, k$, $k\in N$ fixed, be two finite families of nonlinear self mappings of a complete metric space (X, d) with each f_i being continuous. Let f_i and g_i satisfy the following conditions*

(8) $f_ig_j=g_kf_i$, $i\neq j$ *for* $i, j=1, 2, \cdots, k$, $g_i(X)\subset f_i(X)$ *for each i and that a pair of maps f_i and g_j, $i\neq j$, is one to one,*

(9) $$d(g_i(x), g_j(y))\leq a_1d(g_i(x), f_j(x))+a_3d(g_j(y), f_i(y)) +a_3d(g_i(x), f_i(y))+a_4d(g_j(y), f_j(x)) +a_5d(f_j(x), f_i(y))$$

for all $x, y\in X$, $i, j=1, 2, \cdots, k$ with $i\neq j$, where $a_p\geq 0$, $p=1, \cdots, 5$ with $\sum_{p=1}^{5}a_p+a_3<1$. Then all the mappings of both the families $\{f_i\}$ and $\{g_i\}$ have a unique common fixed point in X.

PROOF. Let g_1 and f_2 are one to one. Consider f_1, g_1 and f_2, g_2 the four mappings satisfying (8) and (9), then by Theorem 1 we have a unique common fixed point for this set of four mappings. Let the fixed point be $g_1(t)$. Similarly taking f_1, g_1 and f_3, g_3 to be the next set of four mappings satisfying (8) and (9), we get $g_1(s)$ (say) as the unique common fixed point of f_1, g_1 and f_3, g_3 by Theorem 1. Now $g_1f_2=f_2g_1$ gives that

or,
$$g_1(f_2(g_1(s)))=f_2(g_1(g_1(s)))$$
$$g_1(g_1(f_2(s)))=f_2(g_1(s))=g_1(f_2(s))$$

since g_1 is one we therefore have
$$g_1(f_2(s))=f_2(s)$$
i.e., $f_2(s)$ is also a fixed point of g_1. Then by the uniqueness of the common fixed point of f_1, g_1, g_2, f_2 we have $f_2(s)=f_2(t)$ (from (5)). Then it follows from f_2 being one to one that $s=t$ and thus both the sets of four mappings have a unique common fixed point, viz., $f_i(t)=g_j(t)$, $i, j=1, 2, 3$. Similarly for every pair of four mappings f_1, g_1, f_2, g_2 and f_1, g_1, f_i, g_i where f_i and g_i vary over the rest of the members of the two families f_i and g_i of mappings respectively, we get the same unique common fixed point and hence the result.

In a compact metric space we have the following result.

THEOREM 4. *Let $\{f_i\}$ and $\{g_i\}$, $i=1, 2, \cdots, k$, $k\in N$ fixed, be two finite families of nonlinear self mappings of a compact metric space (X, d) with each f_i being continuous. Let f_i and g_i satisfy the following conditions.*

(10) *$f_ig_j=g_jf_i$, $i\neq j$ for $i, j=1, 2, \cdots, k$, $g_i(X)\subset f_i(X)$ for each i and that a pair of maps f_i and g_j, $i\neq j$ is one to one.*

(11) $$d(g_i(x), g_j(y))\leq a_1d(g_i(x), f_j(x))+a_2d(g_j(y), f_i(y)) +a_3d(g_i(x), f_i(y))+a_4d(g_j(y), f_j(x)) +a_5d(f_j(x), f_i(y))$$

for all $x, y\in X$, $i, j=1, 2, \cdots, k$ with $i\neq j$, where $a_p\geq 0$ with $\sum_{p=1}^{5}a_p+a_2=1$. Then all the mappings of both the families $\{f_i\}$ and $\{g_i\}$ have a unique common fixed point in X.

2. In this section we generalize Theorem 3 and theorem 4 of Mukherjee [2] for common fixed point of four mappings on a Hausdorffspace and on a compact metric space respectively. Further the results have been generalized for two finite families of continuous mappings.

THEOREM 5. *Let for $i=1, 2$ f_i and g_i be four continuous mappings of a Hausdorff space X into itself. Let $f_1g_2=g_2f_1$, $f_2g_1=g_1f_2$ and $g_i(X)\subset f_i(X)$ for each i. Let $F: X\times X\longrightarrow R^+$ be a continuous function satisfying*

(12) $$F(g_1(x), g_2(y))\leq\frac{\alpha F(f_1(y), g_2(y))[1+F(f_2(x), g_1(x))]}{1+F(f_2(x), f_1(y))}$$

$$+\beta F(f_2(x),\ f_1(y))$$

for all $x,\ y\in X$, *where* $\alpha,\ \beta\geq 0$ *with* $\alpha+\beta<1$. *And* $F(u,\ u)=0$ *for all* $u\in X$. *For any* $x_0\in X$, *we define a sequence* $\{y_n\}$ *as follows*

Let $x_1\in X$ *be such that* $g_1(x_0)=f_1(x_1)=y_1$, *and similarly* $x_2\in X$ *be such that* $g_2(x_1)=f_2(x_2)=y_2$, *and so on.*

In general $g_1(x_{2n})=f_1(x_{2n+1})=y_{2n+1}$,

$$g_2(x_{2n+1})=f_2(x_{2n+2})=y_{2n+2},\ n=0,\ 1,\ \cdots.$$

If the sequence $\{y_n\}$ *has a convergent subsequence of the type* $\{y_{n_k+p}\}$, *where* $p=0$, 1 *and* 2, *converging to some* $t\in X$ *for each* p, *then* $f_i(t)=g_j(t)$, $i,\ j=1,\ 2$ *is the unique common fixed point of all* f_i *and* g_i.

PROOF. By omitting point and relabelling if necessary, we may suppose that either all the n_ks are all even or all odd. Let us suppose that each n_k is even. Denoting n_k by m, we have

$$f_1(x_m)\longrightarrow t,\ f_2(x_{m+1})\longrightarrow t,\ f_1(x_{m+2})\longrightarrow t \text{ and}$$
$$g_1(x_{m-1})\longrightarrow t,\ g_2(x_m)\longrightarrow t \text{ and } g_1(x_{m+1})\longrightarrow t.$$

Now, since $f_i,\ g_j,\ i\neq j$ for $i,\ j=1,\ 2$ commute, we get

$$f_1(g_2(x_m))=g_2(f_1(x_m)) \text{ and } f_2(g_1(x_{m+1}))=g_1(f_2(x_{m+1})).$$

Taking limits of the above as each f_i and g_i is continuous, we have

$$f_1(t)=g_2(t) \text{ and } f_2(t)=g_1(t)$$

We claim that $g_1(t)=g_2(t)$. *For if it is not so,* then by (12), we have

$$F(g_1(t),\ g_2(t))\leq\frac{\alpha F(f_1(t),\ g_2(t))\,[1+F(f_2(t),\ g_1(t))]}{1+F(f_2(t),\ f_1(t))}$$
$$+\beta F(f_2(t),\ f_1(t))$$

or, $(1-\beta)F(g_1(t),\ g_2(t))\leq 0$ but $1-\beta>0$, therefore $F(g_1(t),\ g_2(t))=0$ i.e. $g_1(t)=g_2(t)$, and hence

(13) $f_i(t)=g_j(t),\ i,\ j=1,\ 2,\ \cdots.$

Now

$$F(g_1(g_1(t)),\ g_2(t))\leq\frac{\alpha F(f_1(t),\ g_2(t))\,[1+F(f_2(g_1(t)),\ g_1(g_1(t)))]}{1+F(f_2(g_1(t)),\ f_1(t))}$$
$$+\beta F(f_2(g_1(t)),\ f_1(t))$$

or, $(1-\beta)F(g_1(g_1(t)),\ g_1(t))\leq 0$, but $(1-\beta)>0$, therefore $g_1(g_1(t))=g_1(t)$. Further

$$F(g_1(t),\ g_2(g_1(t)))\leq\frac{\alpha F(f_1(g_1(t)),\ g_2(g_1(t)))\,[1+F(f_2(t),\ g_1(t))]}{1+F(f_2(t),\ f_1(g_1(t)))}$$
$$+\beta F(f_2(t),\ f_1(g_1(t)))$$

Using $f_1(g_1(t))=f_1(g_2(t))=g_2(f_1(t))=g_2(g_1(t))$ and $f_2(t)=g_1(t)$ in the above, we get

$$(1-\beta)\ F(g_1(t),\ g_2(g_1(t)))\leq 0$$

and therefore $g_2(g_1(t))=g_1(t)$, since $\beta<1$.

New,

$$f_1(g_1(t))=f_1(g_2(t))=g_2(f_1(t))=g_2(g_1(t))=g_1(t)$$

and $$f_2(g_1(t))=g_1(f_2(t))=g_1(g_1(t))=g_1(t).$$

Thus $g_1(t)$ is the common fixed point of f_1, f_2, g_1, g_2 and therefore from (13) $f_i(t)=g_j(t)$, i, $j=1$, 2 is a common fixed point of f_1, f_2, g_1, g_2. To show uniqueness let $f_i(t)=g_j(t)=z$ (say) and $f_i(s)=g_j(s)=q$(say) be two common fixed points of all f_i and g_i. Then from (12) we have

$$F(g_1(z),\ g_2(q))\leq\frac{\alpha F(f_1(q),\ g_2(q))\,[1+F(f_2(z),\ g_1(z))]}{1+F(f_2(z),\ f_1(q))}+\beta F(f_2(z),\ f_1(q))$$

or, $F(z,\ q)\leq\beta F(z,\ q)$

Since $\beta<1$, we get that $z=q$ and hence the result.

In a compact metric space we have the following result.

THEOREM 6. *Let f_i and g_i, $i=1$, 2, be four continuous mappings of a compact metric space $(X,\ d)$ into itself. Let $f_1g_2=g_2f_1$, $f_2g_1=g_1f_2$ and $g_i(X)\subset f_i(X)$ for each i. Let f_i and g_i satisfy*

$$d(g_1(x),\ g_2(y))\leq\frac{\alpha d(f_1(y),\ g_2(y))\,[1+d(f_2(x),\ g_1(x))]}{1+d(f_2(x),\ f_1(y))}+\beta d(f_2(x),\ f_1(y))$$

for all x, $y\in X$, where α, $\beta\geq 0$ with $\alpha+\beta=1$. Then all f_i and g_i have a unique common fixed point in X.

REMARK 3. If we take $f_1=f_2$ and $g_1=g_2$ in Theorems 5 and 6, we get Theorems 3 and 4 or Mukherjee [2] respectively as our corollaries.

THEOREM 7. *Let for $i=1,\ 2,\ \cdots,\ k$, $k\in N$ fixed, $\{f_i\}$ and $\{g_i\}$ be two finite families of continuous mappings on a Hausdorff space X into itself. Let $f_ig_j=g_jf_i$, $i\neq j$ for i, $j=1,\ 2,\ \cdots$. k, and $g_i(X)\subset f_i(X)$ for each i. Let $F: X\times X\longrightarrow R^+$ be a continuous function satisfying*

$$F(g_i(x),\ g_j(y))\leq\frac{\alpha F(f_i(y),\ g_j(y))\,[1+F(f_j(x),\ g_i(x))]}{1+F(f_j(x),\ f_i(y))}$$

$$+\beta F(f_j(x),\ f_i(y))$$

for all $x,\ y\in X$ *and for each pair* (i, j) *with* $i\neq j$, *where* $\alpha,\ \beta\geq 0$ *with* $\alpha+\beta<1$, *and* $F(u,\ u)=0$ *for all* $u\in X$. *For any* $x_0\in X$ *and for each pair* $(i,\ j)$ *with* $i\neq j$, *define a sequence* $\{y_n\}$ *as follows let* $x_1\in X$ *be such that* $g_i(x_0)=f_i(x\ )=y_1$, *let* $x_2\in X$ *be such that* $g_j(x_1)=f_j(x_2)=y_2$ *and so on.*

In general $g_i(x_{2n})=f_i(x_{2n+1})=y_{2n+1}$,

$$g_j(x_{2n+1})=f_j(x_{2n+2})=y_{2n+2},\ n=0,\ 1,\ \cdots.$$

If this sequence $\{y_n\}$ *has a convergent subsequence of the type* $\{y_{n_k+p}\}$, *where* $p=0$, 1 *and* 2, *converging to some* $t\in X$ *for each* p, *and if a pair of maps* f_i *and* g_j, $i\neq j$, *is one to one, then* $f_i(t)=g_j(t)$, $i,\ j=1,\ 2,\ \cdots,\ k$ *is the unique common fixed point of all* f_i *and* g_i.

PROOF. The proof of the above theorem follows from that of Theorem 5 and Theorem 3.

In a compact metric space $(X,\ d)$ we have the following result.

THEOREM. 8. *Let for* $i=1,\ 2,\ \cdots,\ k,\ k\in N$ *fixed,* $\{f_i\}$ *and* $\{g_i\}$ *be two finite families of continuous mappings on a compact metric space* $(X,\ d)$ *into itself. Let* $f_ig_j=g_jf_i$, $i\neq j$ *for* $i,\ j=1,\ 2,\ \cdots.\ k$, *and* $g_i(X)\subset f_i(X)$ *for each* i. *Let for each pair* $(i,\ j)$ *with* $i\neq j$ f_i *and* g_j *satisfy*

$$d(g_i(x),\ g_j(y)\leq\frac{\alpha\, d(f_i(y),\ g_j(y))\,[1+d(f_j(x),\ g_i(x))]}{1+d(f_j(x),\ f_i(y))}$$
$$+\beta\, d(f_j(x),\ f_i(y))$$

for all $x,y\in X$, *where* $\alpha,\ \beta\geq 0$ *with* $\alpha+\beta=1$. *Further if a pair of maps* f_i *and* g_j, $i\neq j$, *is one to one then all* f_i *and* g_i *have a unique common fixed point in* X.

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