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A NOTE ON H-SETS

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Abstract: The nature of a H-set in a Hausdorff space is not well understood. In this note it is shown that if X is a countable union of nowhere dense compact sets, then X is not H-embeddable in any Hausdorff space. An example is given to show that there exists a non-Urysohn, non-H-closed space X such that each H-set of X is compact.

Introduction.

All spaces under consideration are Hausdorff. Let X be a space, and A a subset of X, then $\mathrm{cl}_X A$ denotes the closure of A in X, $\tau(X)$ denotes the topology and $|X|$ denotes the cardinality of X. An open filter on X is a filter each of whose members is an open subset of X. If $\mathscr{F}$ is a filter on X, then $\mathrm{ad}_X(\mathscr{F})=\bigcap\{\mathrm{cl}_X(F):F\in\mathscr{F}\}$ is called the *adherence* of $\mathscr{F}$: $\mathscr{F}$ is called *free* if $\mathrm{ad}_X(\mathscr{F})=\emptyset$. As pace X is called *H-closed* if X is closed in every space Z in which X is embedded. X is called *minimal Hausdorff* if X has no Hausdorff topology strictly coarser than $\tau(X)$. X is called *Urysohn* if any two distinct points of X are contained in disjoint closed neighborhoods. A subset A of a space X of a space X is *θ-closed* [6] if $A=\{x\in X$: every closed neighborhood of x meets $A\}$. A subset A of a space X is called an *H-set* (see [4], [6]) if whenever $\mathscr{G}\subseteq\tau(X)$ is any covering of A then there exist finitely many $G_1, G_2, \cdots, G_n\in\mathscr{G}$ such that $A\subseteq\bigcup_{i=1}^{n}\mathrm{cl}_X(G_i)$. Every H-closed subset of a space X is an H-set in X and every H-set of X is closed, but the converse need not be true (see [4]). It can be easily shown that a regular closed subset A of a space X is H-closed if and only if A is an H-set. It is remarked in [7] that a subset $A\subseteq X$ is an H-set if and only if whenever $\mathscr{F}$ is an open filter on X such that $A\cap F\neq\emptyset$ for each $F\in\mathscr{F}$, then $A\cap\mathrm{ad}_X(\mathscr{F})\neq\emptyset$. The nature of H-sets in a space is obscure and not well understood. In fact, H-sets behave mysteroiusly in a Hausdorff space. A space X is called *C-compact* [8] if every closed subset of X is a H-set. Katětov [3] proved that a space X is compact if and only if every closed subset of X is H-closed. Viglino [8] gave an example of a noncompact space X such that every closed subset of X is a H-set. It is also proved in [8] that every compact space is C-compact and every C-compact space is minimal Hausdorff, and, moreover, for an Urysohn space all these three notions are equivalent. It is thus pertinent to give an example of a nonregular, non-Urysohn, non-H-closed space X such that each H-set in X is compact. Recall that a map $f: X \to Y$ is θ-continuous if for every x in X and every neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(\mathrm{cl}_X(U)) \subseteq \mathrm{cl}_Y(V)$. Two spaces X and Y are called *θ-homeomorphic* if there exists a bijection f from X onto Y such that f and $f^{\leftarrow}$ are both θ-continuous. We shall call a space X *H-embeddable* if X is a H-set in some Hausdorff space Y.

PROPOSITION 1. *Let f be a θ-continuous map from a space X to a space Y. Then the following are true.*

(a) (see [1]) If A is an H-set in X, then $f(A)$ is an H-set in Y.

(b) If X is H-closed and Y is Urysohn and B is an H-set in Y, then $f^{\leftarrow}(B)$ is an H-set in X.

PROOF. For (a) see [1; 2.5]. To prove (b), let B be an H-set in Y. Since Y is Urysohn, by [2; 2.8], B is θ-closed in Y. New let $x \in X \setminus f^{\leftarrow}(B)$. Then $f(x) \notin B$ and there exists an open neighborhood V of $f(x)$ in Y such that $\mathrm{cl}_Y(V) \cap B = \emptyset$. Since f is θ-continuous, there is an open neighborhood U of x in X such that $f(\mathrm{cl}_X(U)) \subseteq \mathrm{cl}_Y(V)$. Hence, $f(\mathrm{cl}_X(U)) \cap B = \emptyset$, so that $\mathrm{cl}_X(U) \cap f^{\leftarrow}(B) = \emptyset$. Thus. $f^{\leftarrow}(B)$ is θ-closed in X and hence an H-set in X (see [6].)

COROLLARY. *If $f: X \to Y$ is a θ-homeomorphism, then A is an H-set in X if and only if $f(A)$ is an H-set in Y.*

PROPOSITION 2. *If X is a countable union of compact nowhere dense sets, then X is not H-embeddable in any space.*

PROOF. Suppose that X is a H-set in some Hausdorff space Y. Let $X = \bigcup_{n=0}^{\infty} A_n$, where each A_n is compact and nowhere dense in X. Then each A_n is compact in Y. Let $p_0 \in X \setminus A_0$. Since A_0 is compact, there exists an open neighborhood U_1 of p_0 in Y such that $\mathrm{cl}_Y(U_1) \cap A_0 = \emptyset$. Since A_1 is nowhere dense, $(U_1 \cap X) \setminus A_1 \neq \emptyset$. Let $p_1 \in (U_1 \cap X) \setminus A_1$. Since A_1 is compact, there is an open neighborhood U_2 of p_1 in Y with $U_2 \subseteq U_1$ and $\mathrm{cl}_Y(U_2) \subset A_1 = \emptyset$. Assume that we have chosen $U_1 \supseteq U_2$

$\supseteq \cdots\cdots \supseteq U_n$ such that $\mathrm{cl}_Y(U_n) \cap A_{n-1} = \emptyset$ and $U_n \cap X \neq \emptyset$. Since A_n is nowhere dense, $(U_n \cap X) \setminus A_n \neq \emptyset$. Select $p_n \in (U_n \cap X) \setminus A_n$. By compactness of A_n we get an open neighborhood U_{n+1} of p_n in Y with $U_{n+1} \subseteq U_n$ and $\mathrm{cl}_Y(U_{n+1}) \cap A_n = \emptyset$. By induction, we obtain a chain of nonempty open subsets $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots\cdots$ of Y such that $\mathrm{cl}_Y(U_{n+1}) \cap A_n = \emptyset$ and $U_{n+1} \cap X \neq \emptyset$ for all $n = 0, 1, 2 \cdots\cdots$ Let $\mathscr{F}$ be the open filter on Y generated by the family $\{U_i : i = 1, 2, 3 \cdots\}$. Then $F \cap X \neq \emptyset$ for all $F \in \mathscr{F}$. Since X is a H-set in Y, $X \cap \bigcap \{\mathrm{cl}_Y(F) : F \in \mathscr{F}\} \neq \emptyset$. But $\bigcap \{\mathrm{cl}_Y(F) : F \in \mathscr{F}\} \subseteq \bigcup_{n=1}^{\infty} \{\mathrm{cl}_Y(U_n) : n = 1, 2, \cdots\cdots\}$, and since $(\bigcup_{n=1}^{\infty} \mathrm{cl}_Y(U_n)) \cap A_m = \emptyset$ for all $m = 0, 1, 2, \cdots\cdots$, it follows that $X \cap (\bigcup_{n=1}^{\infty} \mathrm{cl}_Y(U_n)) = (\bigcup_{m=0}^{\infty} A_m) \cup (\bigcap_{n=1}^{\infty} \mathrm{cl}_Y(U_n)) = \emptyset$, leading to a contradiction, and the proposition follows.

COROLLARY. (*a*) [7 ; 3.3.13.] *The space* $\boldsymbol{Q}$ *of rationals is not H-embeddable.*

(*b*) *For any non-empty compact space* K, $K \times \boldsymbol{Q}$ *is not H-embeddable.*

(*c*) *Any countable space without isolated points is not H-embeddable.*

The following example shows that there exists a non-regular, non-Urysohn, non-H-closed space X such that each H-set in X is compact. $\boldsymbol{N}$ will denote the set of positive integers.

EXAMPLE. Let $X = \{(1/n, 1/m) : n \in \boldsymbol{N},\ |m| \in \boldsymbol{N}\} \cup \{1/n, 0) : n \in \boldsymbol{N}\} \cup \{0, 1), (0, -1)\}$. Let $\mathscr{U} \in \beta\boldsymbol{N} \setminus \boldsymbol{N}$. Topologize X as follows: a set $U \subseteq X$ is open in X if and only if $U \cap \{X \setminus \{(0, 1), (0, -1)\}\}$ is open in the topology induced by the usual topology of the plane $\boldsymbol{R}^2$ and if $(0, 1) \in U$ (respectively, $(0, -1) \in U$) then there exists a set $K \in \mathscr{U}$ such that $\{(1/n, 1/m) : n \in K, m \in \boldsymbol{N}\} \subseteq U$ (respectively, $\{(1/n, -1/m) : n \in K, m \in \boldsymbol{N}\} \subseteq U$.)

Obviously, X is Hausdorff. Since $(0, 1)$ and $(0, -1)$ cannot be separated by disjoint closed neighborhoods, X is not Urysohn, and hence, X is not regular. We show that each proper H-set of X is compact.

First note that each of the following subsets

$A_n = \{(1/n, 0)\} \cup \{(1/n, 1/m) : m \in \boldsymbol{N}\}$,

$B_n = \{(1/n, 0)\} \cup \{(1/n, -1/m) : m \in \boldsymbol{N}\}$, and

$C_n = \{(1/n, 0)\} \cup \{(1/n, 1/m) : |m| \in \boldsymbol{N}\}$

is a clopen and compact subset of X for each $n \in \boldsymbol{N}$. Also, each point of $X \setminus [\{(0, 1), (0, -1)\} \cup \{(1/n, 0) : n \in \boldsymbol{N}\}]$ is isolated in X. Now, let S be any proper H-set in X. We consider several cases.

Case(1). If $S \cap [\{(0, 1), (0, -1)\} \cup \{(1/n, 0) : n \in \boldsymbol{N}\}] = \emptyset$, then S must be finite and hence compact.

Case(2). If $S=(\bigcup_{i\in I_1} A_i)\cup(\bigcup_{j\in I_2} B_j)\cup(\bigcup_{k\in I_3} C_k)$, then $|I_1|+|I_2|+|I_3|<\aleph_0$. So, again, S is compact.

Case(3). Let $S=\{(0,1)\}\cup\{(1/n,0):n\in P\subseteq N\}$. If $P\notin\mathscr{U}$, then $N\backslash P\in\mathscr{U}$, since $\mathscr{U}$ is an ultrafilter on N. Then $\{(0,1)\}\cup\{(1/n,1/m):n\in N\backslash P,\ m\in N\}\cup\bigcup_{p\in p} A_p$ is an open cover of S. Since each A_p is clopen, then S being a H-set forces the conclusion that $|P|<\aleph_0$. So, S is compact. Now, if $P\in\mathscr{U}$, and P is infinite, let $P=P_1\cup P_2$, where both P_1 and P_2 and infinite and $P_1\cap P_2=\emptyset$. Now either $P_1\in\mathscr{U}$ or $P_2\in\mathscr{U}$. Assume that $P_1\in\mathscr{U}$. Then $(0,1)\cup\{1/n,1/m):n\in P_1,\ m\in N\}\cup\bigcup_{p\in p} A_p$ is an open cover of S. Since S is a H-set, and since $\mathrm{cl}_X[\{(0,1)\}\cup\{(1/n,1/m):n\in P_1,\ m\in N\}]$ misses $\bigcup_{p\in N\backslash p} A_p$, it follows that $|N\backslash P_1|<\aleph_0$, whence $|P_2|\leq|N\backslash P_1|<\aleph_0$, contradicting the assumption that P is infinite. Hence, again, $|P|<\aleph_0$ and S is compact.

Case(4). The case when $S=\{(0,-1)\}\cup\{(1/n,0):n\in P\subseteq N\}$ is handled in the same manner as case(3), and a similar argument then shows that if $S=\{(0,1),(0,-1)\}\cap\{(1/n,0):n\in P\subseteq N\}$ is a H-set, then $|P|<\aleph_0$ and S is compact.

Case(5). Let $S=\{(0,1)\}\cup\{(1/n,0):n\in P\subseteq N\}\cup\bigcup_{q\in Q} A_q$. We show that if S is a H-set, then both P and Q are finite. Assume that P is infinite. If $P\notin\mathscr{U}$ then $N\backslash P\in\mathscr{U}$. So, $\{\{(0,1)\}\cup\{(1/n,1/m):n\in N\backslash P, m\in N\}\}\cup\bigcup_{p\in p} A_p\cup\bigcup_{q\in Q} A_q$ is an open cover of S which does not contain any finite subfamily whose closures cover S (since A_p's are clopen), contradicting the fact that S is a H-set. If $P\in\mathscr{U}$, we let $P=P_1\cup P_2$ as in case(3), and using the same arguments as in case(3) we finally conclude that P is not infinite. Now assume that Q is infinite. If $Q\in\mathscr{U}$ then $N\backslash Q\in\mathscr{U}$. So, $\{\{(0,1)\}\cup\{(1/n,1/m):n\in N\backslash Q, m\in N\}\}\cup\bigcup_{r\in N\backslash Q} A_r\cup\bigcup_{p\in p} A_p$ is an open cover of S which does not contain any finite subfamily whose closures cover S, contradicting the hypothesis that S is an H-set. If $Q\in\mathscr{U}$, we decompose $Q=Q_1\cup Q_2$ as a disjoint union of two infinite subsets, A reasoning similar to the one given in case(3) shows that Q is finite. Thut both P and Q are finite and, hence, S is compact.

A similar reasoning leads to the same conclusion if S contains $(0,-1)$ and/or $\bigcup_{r\in R} B_r$ (respectively, $\bigcup_{d\in D} C_d$). The most general case now follows by taking combinations of the cases (1) through (5).

In the above example it is easy to see that if H_1 and H_2 are two disjoint H-sets in X, then there exist two disjoint open subsets U_1 and U_2 of X containing

H_1 and H_2 respectively and such that $\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) = \emptyset$. Moreover, the topology of X contains a coarser H-closed topology τ', where, a subset U of X is in τ' provided that $U \setminus \{(0,1), (0, -1)\}$ is open in the usual topology of R_2 and if $(0,1) \in U$ (respectively, $(0,-1) \in U$) then there ewists $n_0 \in N$ such that $\{(1/n, 1/m) : n \geq n_0, m \in N\} \subseteq U$ (respectively, $\{(1/n, -1/m) : n \geq n_0, m \in N \subseteq U)$. The space (X, τ') was first defined by Urysohn [5].

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