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# ON A NOTION OF PAIRWISE FUNCTIONAL COMPACTNESS IN BITOPOLOGICAL SPACES

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**Abstract**: In this paper we give a notion of pairwise functional compactness in bitopological spaces. We characterize this class of spaces in terms of continuous functions and covers. Also, a characterization in terms of multifuncions is stated.

## Introduction

Functionally compact Hausdorff spaces were introduced by Dick mann and Zame in 1969 ([2]). These spaces are between $C$-compact spaces, introduced by G. Viglino in 1969, and the well known minimal Hausdorff spaces. Bitopological versions of the two classes of spaces above have just been given and studied in [4], [6], [7]. In this paper we introduce the notion of pairwise functionally compact bitopological spaces. We characterize this class of space by covers and prove that it is between the class of pairwise $C$-compact spaces, in the sense of Swart, and the class of minimal pairwise Hausdorff spaces.

One of the most interesting results obtained by Dickmann and Zame is the following: A Hausdorff spaces $X$ is functionally compact if and only if every continuous mapping from $X$ into a Hausdorff space is closed. At the end of section 2 we prove that our spaces have a similar property (see Theorem 2.2). In section 3 we give some theorems relating pairwise functionally compact bitopological spaces and multifunctions, in particular a generalization of Theorem 2.2 in terms of multifunctions is stated.

## 1. Preliminaries

For general definitions and properties about bitopological spaces, we refer to [3]. In the text, if $(X, \tau_1, \tau_2)$ is a bitopological space, $\mathrm{int}_{\tau_i}$ (resp. $\mathrm{cl}_{\tau_i}$) denotes the interior (resp. closure) taken with respect to $\tau_i$, $i=1, 2$.

We only recall the following

DEFINITION 1.1. A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise Hausdorff if for every pair of points $x, y \in X$, $x \neq y$ there exists a $\tau_i$-open set $U$ and a $\tau_j$-open set $V$ satisfying $x \in U$, $y \in V$, $U \cap V = \emptyset$ for $i \neq j$, $i, j = 1, 2$.

Althought there exist two different notions ([6], [7]) of pairwise $C$-compactness in bitopological spaces, we use the following due to [6].

DEFINITION 1.2. A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise $C$-compact if for every $\tau_i$-closed set $C \subseteq X$ and for every $\tau_j$-open cover $\mathscr{U}$ of $C$ there exists a finite subfamily $\{U_1, \cdots, U_n\}$ of $\mathscr{U}$ such that $C \subseteq \mathrm{Cl}_{\tau_i} U_1 \cup \cdots \cup \mathrm{Cl}_{\tau_i} U_n$, $i \neq j$, $i, j = 1, 2$.

DEFINITION 1.3 [4]. A bitopological space $(X, \tau_1, \tau_2)$ is said to be minimal pairwise Hausdorff if it is pairwise Hausdorff if it is pairwise Hausdorff and if $(X, \phi_1, \phi_2)$ is pairwise Hausdorff with $\phi_1 \subset \tau_1$, $\phi_2 \subset \tau_2$ then $\phi_1 = \tau_1$ and $\phi_2 = \tau_2$.

## 2. Main results

The notion of functionally compact spaces introduced in 1969 by Dickman and Zame [2] is:

DEFINITION 2.1. A Hausdorff topological space $X$ is called functionally compact if, whenever $\mathscr{U}$ is an open filter base on $X$ such that the intersection $A$ of the elements of $\mathscr{U}$ is equal to the intersection of the closures of the elements of $\mathscr{U}$, then $\mathscr{U}$ is a base for the neighborhoods of $A$.

Our purpose is to give a notion of pairwise functional compactness in such way that the class of pairwise functionally compact Hausdorff bitopological spaces is between the class of pairwise $C$-compact Hausdorff bitopological spaces and the class of minimal pairwise Hausdorff bitopological spaces.

DEFINITION 2.2 Let $(X, \tau_1, \tau_2)$ a bitopological space. A subset $C \subseteq X$ is said to be $\theta$-closed if for every $x \in X - C$ there exists a $\tau_i$-open set $U$ such that $x \in U$ and $\mathrm{Cl}_{\tau_j} U \cap C = \emptyset$, $i \neq j$, $i, j = 1, 2$. *A* subset $A$ is said to be $\theta$-open if $X - A$ is $\theta$-closed.

We remark that the above definition is the natural extension to bitopological spaces of the notion of $\theta$-closed set introduced by Velichko in 1968 [8].

DEFINITION 2.3. A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise functionally compact if for every $\tau_i$-open filterbase $\mathscr{U}$, such that $A=\bigcap_{U\in\mathscr{U}} U=\bigcap_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j} U$, $i\neq j$, $i, j=1,2$ and $A$ is $\theta$-closed, $\mathscr{U}$ is a base for the $\tau_i$-neighborhoods of $A$.

Now we give a characterization of pairwise functionally compact spaces in terms of covers.

THEOREM 2.1 *A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is pairwise functionally compact if and only if for every $\tau_j$-closed set $C\subseteq X$ and for every $\tau_i$-open cover $\mathscr{U}$ such that $A=\bigcup_{U\in\mathscr{U}} U=\bigcup_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j} U$ and $A$ is a $\theta$-open set, there exist $U_1, \cdots, U_n$ in $\mathscr{U}$ such that $C\subseteq \mathrm{Cl}_{\tau_j} U_1\cup\cdots\cup\mathrm{Cl}_{\tau_j} U_n$, $i\neq j$, $i, j=1,2$.*

PROOF. Necessity. Let $C$ be a $\tau_j$-closed set and $\mathscr{U}$ a $\tau_i$-open cover of $C$ such that $\bigcup_{U\in\mathscr{U}} U=\bigcup_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j} U$ and this union is $\theta$-open. The family of sets $\{X-(\mathrm{Cl}_{\tau_j} U)\mid U\in\mathscr{U}\}$ generates a $\tau_j$-open filterbase $\mathscr{G}$. We have:

$$\begin{aligned} B=\bigcap_{U\in\mathscr{U}}(X-\mathrm{Cl}_{\tau_j} U)&\subseteq\bigcap_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_i}(X-\mathrm{Cl}_{\tau_j} U)\\ &=\bigcap_{U\in\mathscr{U}}(X-\mathrm{int}_{\tau_i}\mathrm{Cl}_{\tau_j} U)\subseteq\bigcap_{U\in\mathscr{U}}(X-U)=X-\bigcup_{U\in\mathscr{U}} U\\ &=X-\bigcup_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j} U=\bigcap_{U\in\mathscr{U}}(X-\mathrm{Cl}_{\tau_j} U). \end{aligned}$$

This means that $B$ is $\theta$-closed, so $\mathscr{G}$ is a base for the $\tau_j$-neighborhoods of $B$. Because $B\subseteq X-C$ and $X-C$ is a $\tau_j$-open there exist $U_1, \cdots, U_h$ of $\mathscr{U}$ such that:

$$X-C\supseteq\bigcap_{h=1}^{n}(X-\mathrm{Cl}_{\tau_j} U_h)=X-\bigcup_{h=1}^{n}\mathrm{Cl}_{\tau_j} U_h$$

then $C\subseteq\bigcup_{h=1}^{n}\mathrm{Cl}_{\tau_j} U_h$.

Sufficiency. Let $\mathscr{U}$ be a $\tau_i$-open filterbase such that $\bigcap_{U\in\mathscr{U}} U=\bigcap_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j} U=A$ is $\theta$-closed. Let $V$ a $\tau_i$-open set such that $V\supseteq A$. Consider the family $\mathscr{G}=\{X-\mathrm{Cl}_{\tau_j} U\mid U\in\mathscr{U}\}$ that is an $\tau_j$-open cover of $X-V$. We have:

$$\begin{aligned} \bigcup_{U\in\mathscr{U}}(X-\mathrm{Cl}_{\tau_j} U)&\subseteq\bigcup_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_i}(X-\mathrm{Cl}_{\tau_j} U)\\ &=\bigcup_{U\in\mathscr{U}}(X-\mathrm{int}_{\tau_i}\mathrm{Cl}_{\tau_j} U)\subseteq\bigcup_{U\in\mathscr{U}}(X-U)=\bigcup_{U\in\mathscr{U}}(X-\mathrm{Cl}_{\tau_j} U), \end{aligned}$$

moreover

$$\bigcup_{U\in\mathscr{U}}(X-\mathrm{Cl}_{\tau_j} U)=X-A$$

is $\theta$-open. Then there exist $U_1, \cdots, U_n$ in $\mathscr{U}$ such that:

$$X-V\subseteq\bigcup_{h=1}^{n}\mathrm{Cl}_{\tau_i}(X-\mathrm{Cl}_{\tau_j} U_h)=\bigcup_{h=1}^{n}(X-\mathrm{int}_{\tau_i}\mathrm{Cl}_{\tau_j} U)\subseteq\bigcup_{h=1}^{n}(X-U_h).$$

So $V\supseteq\bigcap_{h=1}^{n}U_h$.

By Theorem 2.1 and Definition 1.2 it follows:

COROLLARY 2.1. *Every pairwise Hausdorff, pairwise C-compact bitopological space is functionally compact.*

DEFINITION 2.4. A mapping $f:(X, \tau_1, \tau_2)\to(Y, \sigma_1, \sigma_2)$ is said to be continuous (closed) if the induced mappings $f:(X, \tau_1)\to(Y, \sigma_1)$ and $f:(X, \tau_2)\to(Y, \sigma_2)$ are continuous (closed).

LEMMA 2.1. *Let $(X, \tau_1, \tau_2)$ be a bitopological space and $(Y, \sigma_1, \sigma_2)$ a pairwise Hausdorff bitopological space. If $f:X\to Y$ is a continuous mapping then for every $y\in Y$ $f^{-1}(y)$ is $\theta$-closed.*

PROOF. It follows easily from definitions.

THEOREM 2.2. *A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is pairwise functionally compact if and only if for every pairwise Hausdorff bitopological space $(Y, \sigma_1, \sigma_2)$ every $f:X\to Y$ continuous is closed.*

PROOF. Necessity. Let $C$ be a $\tau_i$-closed set in $X$, and $y\notin f(C)$. Because $Y$ is pairwise Hausdorff for every $x$ in $f(C)$ there exist a $\sigma_i$-open $U_x$ and $\sigma_j$-open $V_x$ such that $y\in U_x$, $x\in V_x$ and $U_x\cap V_x=\emptyset$. The family $G=\{f^{-1}(V_x)\,|\,x\neq y\}$ is a $\tau_j$-open cover of $C$. The set $\bigcup_{x\neq y}f^{-1}(V_x)=X-f^{-1}(y)$ is $\theta$-open by Lemma 2.1, moreover for every $x$ in $Y$, $x\neq y$ is for continuity $\mathrm{Cl}_{\tau_i}(f^{-1}(V_x))\subseteq f^{-1}(\mathrm{Cl}_{\sigma_i}V_x)$ and because $y\notin\mathrm{Cl}_{\sigma_i}V_x$ results $\bigcup\mathrm{Cl}_{\tau_i}(f^{-1}(V_x))=X-f^{-1}(y)$. Now by Theorem 2.1 it follows

$$C\subseteq\mathrm{Cl}_{\tau_i}(f^{-1}(V_{x_1}))\cup\cdots\cup\mathrm{Cl}_{\tau_i}(f^{-1}(V_{x_n}))$$

for suitable $x_1, \cdots, x_n\in Y$, $x_i\neq y$, $i=1, \cdots, n$. Then $f(C)\subseteq\mathrm{Cl}_{\sigma_i}V_{x_1}\cup\cdots\cup\mathrm{Cl}_{\sigma_i}V_{x_n}$. Consider $U=U_{x_1}\cap\cdots\cap U_{x_n}$. We have $U\cap f(C)=\emptyset$ and $y\in U$ *so* $f(C)$ is closed.

Sufficiency. Let $X$ be a pairwise Hausdorff bitopological space such that every continuous mapping in every pairwise Hausdorff bitopological space is closed. Let $\mathscr{U}$ be a $\tau_i$-open filterbase such that $\bigcap_{U\in\mathscr{U}}U=\bigcap_{U\in\mathscr{U}}\mathrm{Cl}_{\tau_j}U=A$ is $\theta$-closed. Define on $X$ the partition $X=A\bigcup_{x\notin A}\{x\}$. Let $Y$ the quotient space with respect to induced equivalence relation and $\pi:X\to Y$ the canonical mapping. Topologize $Y$ as follows: system of neighborhoods for $\Pi(A)$ in $\sigma_i$ is $\{\pi(U)\,|\,U\in\mathscr{U}\}$ and for

$x \notin A$ is $\{\pi(V) \mid V\ \tau_i$-neighborhoods of $x\}$; system of neighborhoods for $\pi(A)$ in $\sigma_j$ is $\{\pi(W) \mid W$ is $\tau_j$-open and $A \subseteq W\}$ and for $x \notin A$ is $\{\pi(W) \mid W\ \tau_j$-neighborhoods of $x\}$. Because $A$ is $\theta$-closed, it is easy to verify that $Y$ is pairwise Hausdorff, moreover the canonical mapping is continuous then closed. Let $Z$ a $\tau_i$-neighborhood of A, because $\pi$ is closed there exists a $\sigma_i$-neighborhood $V$ of $\pi(A)$ such that $\pi^{-1}(V) \subseteq Z$. Then there exists $U \in \mathscr{U}$ such that $\pi(U) \subseteq V$, therefore $U \subseteq Z$.

With a slight modification we can restate the above theorem as follows:

THEOREM 2.2. *A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is pairwise functionally compact if and only if for every pairwise Hausdorff bitopological space $(Y, \sigma_1, \sigma_2)$ every $f: X \to Y$ continuous and surjective is closed.*

COROLLARY 2.2. *If a pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is pairwise functionally compact, is minimal pairwise Hausdorff.*

PROOF. If $(X, \sigma_1, \sigma_2)$ is a pairwise Hausdorff bitopological space with $\sigma_i \subset \tau_i$ the identity map $i: (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is continuous, then closed. This means $\tau_1 = \sigma_1$ and $\tau_2 = \sigma_2$.

THEOREM 2.3. *Let $(X, \tau_1, \tau_2)$ be a bitopological space that is pairwise Hausdorff and pairwise functionally compact. Let $(Y, \sigma_1, \sigma_2)$ be a pairwise Hausdorff bitopological space. If $f: X \to Y$ is a continuous surjection then $(Y, \sigma_1, \sigma_2)$ is pairwise functionally compact.*

PROOF. It is enough to prove that every continuous function $g: Y \to Z$ with $(Z, \eta_1, \eta_2)$ pairwise Hausdorff bitopological space, is closed. If $C$ is a $\sigma_i$-closed in $Y$, $f^{-1}(C)$ is a $\tau_i$-closed in $X$. The mapping $g \circ f: X \to Z$ is continuous, then closed, so:

$$g(C) = g \circ f\,(f^{-1}(C))$$

is a $\eta_i$-closed set in $Z$.

## 3. Pairwise functionally compact spaces and multifunctions

In this section we extend some results on multifunctions and functionally compact spaces given in [1]. Our purpose is to give a generalization of Theorem 2.2 to multifunctions.

For notations and general informations on multifunctions see |5|.

DEFINITION 3.1 A multifunction $\varphi:(X, \tau_1, \tau_2)\to(Y, \sigma_1, \sigma_2)$ is said to be open (resp. closed), lower semicontinuous (resp. upper) or continuous if the induced multifunction $\varphi:(X, \tau_1)\to(Y, \sigma_1)$, $\varphi:(X, \tau_2)\to(Y, \sigma_2)$ are open (resp. closed), lower semicontinous (resp. upper) or continuous.

THEOREM 3.1. *Let $(Y, \sigma_1, \sigma_2)$ be a pairwise Hausdorff bitopological space. Y is pairwise functionally compact if and only if for every pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ every open, $\theta$-closed valued multifunction $\varphi: X \to Y$, such that for every $x\in X$*

$$\varphi(x)=\bigcap_{U\in B_i(x)} \mathrm{Cl}\tau_j\varphi(U),$$

*is upper semicontinuous.*

PROOF. Necessity. Let $(Y, \sigma_1, \sigma_2)$ be pairwise functionally compact. For every fixed $x\in X$, because $\varphi$ is open, the family $\{\varphi(U)|U\in B_i(x)\}$ is a $\sigma_i$-open filterbase. It is, from hypothesis:

$$\varphi(x)=\bigcap_{U\in B_i(x)}\varphi(U)=\bigcap_{U\in B_i(x)}\mathrm{Cl}\,\tau_j\varphi(U)$$

and $\varphi(x)$ is $\theta$-closed. Then, from Definition 2.3, $\{\varphi(U)|U\in B_i(x)\}$ is a system of $\sigma_i$-neighborhoods of $\varphi(x)$. Thus for every $\sigma_i$-open $V$ containing $\varphi(x)$ there exists $U\in B_i(x)$ such that $\varphi(U)\subseteq V$, then $\varphi$ is upper semicontinuous.

Sufficiency. It is enough to prove that every continuous function $f: Y\to X$ with $(X, \tau_1, \tau_2)$ pairwise Hausdorff is closed. Without lost of generality we can suppose $f$ surjective. Let $\varphi: X\to Y$ the multifunction defined with $\varphi=f^{-1}$. $f$ is continuous, then from Lemma 2.1, $\varphi$ is $\theta$-closed valued and open, moreover, because $(X, \tau_1, \tau_2)$ is pairwise Hausdorff, it is

$$\begin{aligned}\varphi(x)&\subseteq\bigcap_{U\in B_i(x)}\mathrm{Cl}\,\sigma_j\varphi(U)=\bigcap_{U\in B_i(x)}\mathrm{Cl}\,\sigma_j f^{-1}(U)\\ &\subseteq\bigcap_{U\in B_i(x)} f^{-1}(\mathrm{Cl}\,\tau_j U)=f^{-1}\bigcap_{U\in B_i(x)}\mathrm{Cl}\,\tau_j(U)=f^{-1}(x)=\varphi(w)\end{aligned}$$

then

$$\varphi(x)=\bigcap_{U\in B_i(x)}\mathrm{Cl}\sigma_j\varphi(U).$$

Thus $\varphi$ is upper semicontinuous and $f$ is closed.

REMARK. It is possible to give analogous results of Theorem 3.1 for pairwise $C$-compact space and minimal pairwise Hausdorff spaces.

THEOREM 3.2. *Let $(X, \tau_1, \tau_2)$ be a pairwise Hausdorff bitopological space and Let $(Y, \sigma_1, \sigma^2)$ be a pairwise Hausdorff, pairwise functionally compact space then every open, closed, $\theta$-closed valued multifunction $\varphi$ with $\theta$-closed fibres, $\varphi: X \to Y$, is upper semicontinuous.*

PROOF. From Theorem 3.1 it is enough to show $\varphi(x) = \bigcap_{U \in B_i(x)} \mathrm{Cl}\sigma_j \varphi(U)$ for every $x$ in $X$. It is always true that $\varphi(x) \subseteq \bigcap_{U \in B_i(x)} \mathrm{Cl}\sigma_j \phi(x)$, to prove the converse it is enough to show that: if $y \notin \varphi(x)$ then there exists $U \in B_i(x)$ such that $y \notin \mathrm{Cl}\sigma_j \varphi(U)$. If $y \notin \varphi(x)$, $x \notin \varphi^{-1}(y)$, but $\varphi^{-1}(y)$ is $\theta$-closed then there exists $U\ B_i(x)$ such that $\mathrm{Cl}\,\tau_j U \cap \varphi^{-1}(y) = \emptyset$, then $y \notin \varphi(\mathrm{Cl}\,\tau_j(U))$. Because $\varphi$ is closed it is $\varphi(\mathrm{Cl}\,\tau_j(U)) \supseteq \mathrm{Cl}\sigma_j(\varphi(U))$ thus $y \notin \mathrm{Cl}\,\sigma_j \varphi(U)$.

To obtain our final result we need the following extension of Lemma 2.1.

LEMMA 3.1. *Let $(X, \tau_1, \tau_2)$, $(Y, \sigma_1, \sigma_2)$ be bitopological spaces, $Y$ is pairwise Hausdorff. If $\varphi: X \to Y$ is a continuous, $\theta$-closed valued multifunction the $\varphi^{-1}(y)$ is $\theta$-closed for every $y \in Y$.*

PROOF. Let $y$ be a point in $Y$ and let $x$ be a point in $X$ such that $x \notin \varphi^{-1}(y)$. $\varphi(x)$ is $\theta$-closed then there exists $U \in B_i(y)$ such that $\mathrm{Cl}\,\sigma_j U \cap \varphi(x) = 0$ thus $x \notin \varphi^{-1}(\mathrm{Cl}\,\sigma_j(U))$. Because $\varphi$ is continuous it is:

$$\varphi^{-1}(U) \subseteq \mathrm{Cl}\,\tau_j \varphi^{-1}(U) \subseteq \varphi^{-1}(\mathrm{Cl}\sigma_j(U)),$$

moreover $\varphi^{-1}(y) \subseteq \varphi^{-1}(U)$. This proves the lemma.

THEOREM 3.3 *A pairwise Hausdorff bitopological space $(X, \tau_1, \tau_2)$ is pairwise functionally compact if and only if for every pairwise Hausdorff bitopological space $(Y, \sigma_1, \sigma_2)$, every continuous, surjective, $\theta$-closed valued multifunction $\varphi: X \to Y$ is closed.*

PROOF. Necessity. Define $\phi: Y \to X$ with $\phi = \varphi^{-1}$. Because is surjective, continuous and $\theta$-closed valued $\phi$ is well defined, open, with $\theta$-closed fibres, moreover from Lemma 3.1 is $\theta$-closed valued. Then from Theorem 3.2 $\phi$ is upper semicontinuous thus $\varphi$ is closed.

Sufficiency. Let $f: X \to Y$ be a continuous, surjective function. $f$ is $\theta$-closed valued, because $Y$ is pairwise Hausdorff. Then $f$ is closed. From Theorem 2.2 follows that $X$ is pairwise functionally compact.

## REFERENCES

[1] A. Bella, *A Note on functionally compact spaces*, Proc. Amer. Math. Soc. 1986, to appear.

[2] R.F. Dickman, A. Zame, *Functionally compact spaces*, Pac. J. of Math., 31(1969), 303—311.

[3] J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc., 13(1963), 71—89.

[4] T.G. Raghavan and I.L. Reilly, *On Minimal Bitopological Spaces*, Kyungpook Math. J., 17, n.1(1977), 57—65.

[5] R.E. Smithson, *Multifunctions*, Nieuw Arch. Wisk, (3) 20(1972), 31—53.

[6] J. Swart, *Pairwise C-compact spaces*, Kyungpoook Math. J., 19 n.1 (1979), 33—38.

[7] R. Vasudevan and C.K. Goel, *A note on pairwise C-compact bitopological spaces*, Mat. Bec., 1(14) (29), 1977, 179—187.

[8] N.V. Velicko, *H-closed topological spaces*, Mat. Sb., 70(112) (1966), 98—112=Amer. Math. Soc. Transl. 78(2) (1968), 103—118.

[9] G. Viglino, *C-compact spaces*, Duke J. Math., 36(1969), 761—764.

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