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THE EQUIVALENCE OF COMPACTNESS AND PSEUDO-COMPACTNESS IN SOME FUNCTION SPACES

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Abstract: This paper investigates the relationship between compactness and pseudo-compactness in subsets of $C(X)$ where X is locally compact and first countable. Two primary theorems are proven. First, equicontinuity at a point is proven to be equivalent to the existence of a certain open cover of a pseudo-compact subset of $C(X)$. The second theorem proves the equivalence of compactness and pseudo-compctness for closed subsets F of $C(X)$.

Introduction

Historically, much of the study of function spaces has focused on finding conditions which will assure that a given family of real-valued continuous functions be compact in the compact-open topology. This study has produced the classical Ascoli Theorem [4] and numerous generalizations, for example, [3, 4]. In this paper, we use the Ascoli Theorem to establish the equivalence of compactness and pseudocompactness in $C(X)$ for a large class of domain spaces.

We first establish some notations and definitions.

All spaces X will be Hausdorff, and all functions will belong to $C(X)$. The compact-open (respectively, pointwise) topology on $C(X)$ will be denoted by τ_c(resp., τ_p) and the closure operator will be denoted by Cl_C(resp., Cl_p). A subbasic open set in τ_c will be denoted by (K,U). where $K\subset X$ is compact and $U\subset R$ is open.

If $x\in R$, we denote by $B(x,\varepsilon)$ the open interval of radius $\varepsilon>0$ about x. The rational numbers will be denoted by Q.

Ascoli's Theorem charaterizing compactness in $C(X)$ involves the following notation:

DEFINITION: Let X be a topological space and let $x_0\in X$. Then $F\subset C(X)$ is *equicontinuous at* x_0 if for every $\varepsilon>0$, there is an open set $U\subset X$ containing x_0

such that $f(U) \subset B(f(x_0), \varepsilon)$ for all $f \in F$. Then F is *equicontinuous* if it is equicontinuous at each point of X.

In [5], Michael defined the concept of a cushioned refinement as follows:

DEFINITION. A collection $\{A_\alpha | \alpha \in \mathscr{A}\}$ is *cushioned* in $\{B_\alpha | \alpha \in \mathscr{A}\}$ if for each $\mathscr{B} \subset \mathscr{A}$, $\mathrm{Cl}(\cup \{A_\beta | \beta \in \mathscr{B}\}) \subset \cup \{B_\beta | \beta \in \mathscr{B}\}$.

Preliminary results

LEMMA 1. *If $\varepsilon < \varepsilon'$, and K is compact, then $\{(K, B(q, \varepsilon)) | q \in Q\}$ is cushioned in $\{(K, B(q, \varepsilon')) | q \in Q\}$.*

PROOF. Assume not. Then there exists $P \subset Q$ and $f \in (C(X), \tau_c)$ such that $f \in \mathrm{Cl}_C(\cup \{(K, B(q, \varepsilon)) | q \in P\}) - \cup \{(K, B(q, \varepsilon') | q \in P\}$.

Let $\{f_\alpha\}$ be a net in $\cup \{(K, B(q, \varepsilon') | q \in P\}$ which converges to f. Set $\varepsilon_0 = \varepsilon' - \varepsilon > 0$. Since $\{f_\alpha\} \to f$, there exists α_0 such that $F_{\alpha_0} \in (K, \cup \{B(f(x), \varepsilon_0) | x \in K\})$. Thus, for each $x \in K$, $-\varepsilon_0 < f(x) - f_{\alpha_0}(x) < \varepsilon_0$, or equivalently,

(1) $$f_{\alpha_0}(x) - \varepsilon' + \varepsilon < f(x) < f_{\alpha_0}(x) + \varepsilon' - \varepsilon$$

But since $\{f_\alpha\} \subset \cup \{K, B(q, \varepsilon)) | q \in P\}$, there exists $q_0 \varepsilon P$ such that $f_{\alpha_0} \in (K, B(q_0, \varepsilon))$. Thus for all $x \in K$, $q_0 - \varepsilon < f_{\alpha_0}(x) < q_0 + \varepsilon$. Then by (1), we have for all $x \in K$,

$q_0 - \varepsilon - \varepsilon' + \varepsilon < f(x) < q_0 + \varepsilon + \varepsilon' - \varepsilon$, or $q_0 - \varepsilon' < f(x) < q_0 + \varepsilon'$. Hence $f \in (K, B(q_0, \varepsilon')) \subset \cup \{(K, B(q, \varepsilon')) | q \in P\}$, a contradiction.

To establish our main results, we will need two additional lemmas. The first is a well-known fact about $(C(X), \tau_c)$ and the second is a characterization of pseudocompactness in Tychonoff spaces due to Hewitt [2]. We record them here for future reference.

LEMMA 2. *If $K \subset X$ is compact and $V \subset R$ is open, then* $\mathrm{Cl}_c(K, V) \subset (K, \mathrm{Cl}(V))$.

LEMMA 3. *A Tychonoff space X is pseudocompact if and only if for every decreasing sequence of non-empty open sets $\{U_n\}$ in X,* $\cap \mathrm{Cl}(U_n) \neq \emptyset$.

Principal results

The first result establishes that, under rather general conditions imposed on the domain space, equicontinuity at a point for a pseudocompact collection of functions is equivalent to the existence of a particular τ_c-open cover of the collection.

THEOREM 1. *Let X be a locally compact and first countable and let F be a pseudocompact subset of $((C(X),\tau_c)$. Then F is equicontinuous at x_0 if and only if $\{(K_n, B(q,\varepsilon))\,|\,n\in N,\ q\in Q\}$ is an open cover of F for each $\varepsilon>0$, where $\{K_n\}$ is a neighborhood base of compact sets at x_0.*

PROOF. Assume F is equicontinuous at x_0 and let $\varepsilon>0$ be given. Then there exists a compact neighborhood K_m of x_0 such that $f(K_m)\subset B(f(x_0),\varepsilon/2)$ for all $f\in F$. Hence $\{(K_n, B(q,\varepsilon))\,|\,n\in N,\ q\in O\}$ is an open cover of F.

Conversely, assume F is not equicontinuous at x_0. Then there exists $\varepsilon_0>0$, functions $f_n\in F$, and points $x_n\in X$ such that, for each $n\in N$, $f_n(x_n)\notin B(f_n(x_0),\varepsilon_0)$. This implies that for each $q\in Q$, $f_n\notin(K_n, B(q,\varepsilon_0/2))$ for each n and hence $f_n\notin\cup\{K_n, B(q,\varepsilon_0/2))\,|\,q\in Q\}$. Now let $\varepsilon<\varepsilon_0/2$ and note that by Lemma 1, $f_n\notin\mathrm{Cl}_c(\cup\{K_n, B(q,\varepsilon))\,|\,q\in Q\})$ for each n. Thus $f_n\in(F-Cl_c)\cup\{(K_n, B(q,\varepsilon))\,|\,q\in Q\})$. Since F is pseudocompact and $\{F-\mathrm{Cl}_c(\cup\{(K_n,\ B(q,\varepsilon))\,|\,q\in Q\})\}$ is a decreasing sequence of open sets, we have by Lemma 3 that $\bigcap_n \mathrm{Cl}_c(F-\mathrm{Cl}_c(\cup\{(K_n, B(q,\varepsilon))\,|\,q\in Q\}))\neq\emptyset$, and hence $\bigcap_n \mathrm{Cl}_c(F-\cup\{(K_n, B(q,\varepsilon))\,|\,q\in Q\})\neq\emptyset$. By DeMorgan's Law, it follows that $\bigcap_n \mathrm{Cl}_c(\cap\{F-(K_n, B(q,\varepsilon))\,|\ q\in Q\})\neq\emptyset$.

$$
\begin{aligned}
\text{Thus } F\supset F-&\bigcap_n \mathrm{Cl}_c(\cap\{F-(K_n, B(q,\varepsilon))\,|\,q\in Q\})\\
&=\bigcup_n(F-\mathrm{Cl}_c(\cap\{F-(K_n, B(q,\varepsilon))\,|\,q\in Q\})\\
&\supset\bigcup_n(F-n\{\mathrm{Cl}_c(F-(K_n, B(q,\varepsilon))\,|\,q\in Q\})\\
&=\bigcap_{n,q}(F-\mathrm{Cl}_c(F-(K_n, B(q,\varepsilon))))\\
&=\bigcup_{n,q}(F-(F-(K_n, B(q,\varepsilon))))=\bigcup_{n,q}(K_n, B(q,\varepsilon)).
\end{aligned}
$$

Therefore, $\{(K_n, B(q,\varepsilon))\,|\,n\in N,\ q\in Q\}$ is not an open cover of F.

Our final theorem establishes that, assuming the same conditions on the domain space, for closed subspaces of $C(X)$ with the compact-open topology, compactness and pseudocompactness are equivalent.

THEOREM 2. *Let X be a locally compact and first countable and let F be a closed subset of $(C(X),\tau_c)$. Then F is compact if and only if it is pseudocompact.*

PROOF. We need only show that if F is pseudocomapct, it satisfies the three conditions of Ascoli's Theorem (Kelley, p. 233), from which compactness follows.

We first establish that F is equicontinuous. Let $x_0 \in X$ and let $\{K_n\}$ be a neighborhood base of compact sets at x_0. Given $\varepsilon > 0$, for each $f \in F$, there must exist $q \in Q$ such that $f(x_0) \in B(q, \varepsilon)$. Thus the collection $\{(K_n, B(q, \varepsilon)) \mid n \in N, q \in Q\}$ is a τ_c-open cover of F. Equicontinuity follows from Theorem 1.

Since F is equicontinuous on X, so is $\mathrm{Cl}_p(F)$ by [Kelley, p. 232]. Note that $\mathrm{Cl}_c(F) \subset \mathrm{Cl}_p(F)$, but since $\mathrm{Cl}_p(F)$ is an equicontinuous family, the topologies τ_c and τ_p coincide on this set [Kelley, p. 232], so $\mathrm{Cl}_c(F) = \mathrm{Cl}_p(F)$. But $\mathrm{Cl}_c(F) = F$ by hypothesis, Thus F is pointwise closed.

Finally, we must establish that for each $x \in X$, $\mathrm{Cl}(\{f(x) \mid f \in F\})$ is compact in R. Suppose there exists $x_0 \in X$ such that $\mathrm{Cl}(\{f(x_0) \mid f \in F\})$ is not compact. Then there is a sequence of functions $\{f_n\} \subset F$ such that $\{f_n(x_0) \mid n \in N\}$ is unbounded, say unbounded above without loss of generality. Select a subsequence $\{f_k\}$ of $\{f_n\}$ having the property that $f_k(x_0) > k$ for each $k \in N$. Now consider the neighborhood $V_k = (\{x_0\}, (k, +\infty))$ of f_k for each k. Each V_k is a non-empty open set in the pseudocompact space (F, τ_c), but $\bigcap \{\mathrm{Cl}_c(V_k) \mid k \in N\} \subset \bigcap \{(\{x_0\}, (k, +\infty)) \mid k \in N\} = \emptyset$, which contradicts Lemma 3. Thus $\{f(x) \mid f \in F\}$ has compact closure for each $x \in X$. This completes the proof of the theorem.

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