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ALMOST α-CONTINUOUS FUNCTIONS

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1. Introduction

In 1968, Singal and Singal [14] introduced the notion of almost-continuous functions. Recently, Mashhour et. al. [8] have defined and investigated a new class of functions called α-continuous functions. These functions have been further investigated by Reilly and Vamanamurthy [13] and the present author [11]. On the other hand, Maheshwari et. al. [6] introduced the concept of almost feebly continuous functions.

The purpose of the present paper is to introduce the concept of almost α-continuity in topological spaces as a generalization of α-continuity and almost-continuity. In Section 3, we obtain several characterizations of almost α-continuous functions and show that almost α-continuity is equivalent to almost feeble continuity. In the last section, we obtain several properties of almost α-continuous functions and a characterization of α-irresolute functions due to Maheshwari and Thakur [5].

2. Preliminaries

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of (X, τ). The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. A subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S))=S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S))=S$). A subset S is said to be *α-open* [9] (resp. *semi-open* [4], *pre-open* [8]) if $S\subset\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$ (resp. $S\subset\mathrm{Cl}(\mathrm{Int}(S))$, $S\subset\mathrm{Int}(\mathrm{Cl}(S))$). The complement of an α-open (resp. semi-open) set is called *α-closed* (resp. *semi-closed*). The family of all α-open (resp. semi-open, pre-open) sets of (X, τ) is denoted by τ^{α} (resp. $\mathrm{SO}(X, \tau)$, $\mathrm{PO}(X, \tau)$). It is shown in [9] that τ^{α} is a topology for X and $\tau\subset\tau^{\alpha}\subset\mathrm{SO}(X, \tau)$. It is also shown in [11] that $\tau^{\alpha}=\mathrm{PO}(X, \tau)\cap\mathrm{SO}(X, \tau)$.

DEFINITION 2.1. A function $f:(X, \tau)\to(Y, \sigma)$ is said to be *α-continuous* [8] if $f^{-1}(V)\in\tau^{\alpha}$ for every $V\in\sigma$.

DEFINITION 2.2 A function $f:(X, \tau)\to(Y, \sigma)$ is said to be *almost continuous* [14] if for each $x\in X$ and each $V\in\sigma$ containing $f(x)$, there exists $U\in\tau$ containing x such that $f(U)\subset \mathrm{Int}(\mathrm{Cl}(V))$.

REMARK 2.1. It is shown in [11] that α-continuity and almost-continuity are independent of each other.

DEFINITION 2.3. A function $f:(X, \tau)\to(Y, \sigma)$ is said to be *almost α-continuous* (briefly $a.\alpha.c$) if $f^{-1}(V)\in\tau^{\alpha}$ for every regular open set V of (Y, σ).

REMARK 2.2. It is obvious that almost α-continuity is implied by α-continuity and almost-continuity. However, by Remark 2.1 the converses are not true in general.

DEFINITION 2.4. A function $f:(X, \tau)\to(Y, \sigma)$ is said to be *η-continuous* [2] if every regular open sets U, V of (Y, σ),

(a) $f^{-1}(V)\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ and

(b) $\mathrm{Int}(\mathrm{Cl}(f^{-1}(U\cap V)))=\mathrm{Int}(\mathrm{Cl}(f^{-1}(U)))\cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$.

REMARK 2.3. It is shown in [11] and [2] that both α-continuity and almost-continuity imply η-continuity.

THEOREM 2.1. *If $f:(X, \tau)\to(Y, \sigma)$ is $a.\alpha.c.$, then it is η-continuous.*

PROOF. Let U, V be any regular open sets of (Y, σ). Since f is $a.\alpha.c.$, $f^{-1}(V)\in\tau^{\alpha}\subset \mathrm{PO}(X, \tau)$ and hence $f^{-1}(V)\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$. Moreover, since $f^{-1}(V)\in\tau^{\alpha}\subset \mathrm{SO}(X, \tau)$, by Lemma 3.5 of [11] we obtain (b) of Definition 2.4.

REMARK 2.4. The converse of of Theorem 2.1 is not true in general as the following example shows.

EXAMPLE 2.1. Let $X=\{a, b, c, d\}$ and $\tau=\{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$. Let $Y=\{x, y, z\}$ and $\sigma=\{\emptyset, \{x\}, \{z\}, \{x, z\}, Y\}$. Define a function $f:(X, \tau)\to(Y, \sigma)$ as follows: $f(a)=x$, $f(b)=f(c)=y$ and $f(d)=Z$. Then f is η-continuous but it is not $a.\alpha.c.$ since $\{x\}$ is regular open in (Y, σ) and $\{a\}\notin\tau^{\alpha}$.

REMARK 2.5. We have the following relationships, however, by Remarks 2.1 and 2.4 none of these implications is reversible.

$$\begin{array}{ccccc} & \nearrow \alpha\text{-continuous} & \searrow & & \\ \text{continuous} & & & a.\alpha.c. \longrightarrow \eta\text{-continuous} \\ & \searrow \text{almost-continuous} & \nearrow & & \end{array}$$

3. Characterizations

Let S be a subset of a space (X, τ). The intersection of all semi-closed sets containing S is called the *semi-closure* of S [1] and is denoted by $s\mathrm{Cl}(S)$. A subset S is said to be *feebly open* [7] if there exists $U \in \tau$ such that $U \subset S \subset s\mathrm{Cl}(U)$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost feebly continuous* (resp. *feebly continuous*) [6] if $f^{-1}(V)$ is feebly open in (X, τ) for every regular open (resp. open) set V of (Y, σ).

LEMMA 3.1. *Let U be a subset of a space (X, τ). Then $U \in PO(X, \tau)$ if and only if* $\mathrm{Int}(\mathrm{Cl}(U)) = s\mathrm{Cl}(U)$.

PROOF. Suppose that $U \in \mathrm{PO}(X, \tau)$. It is shown that $\mathrm{Int}(\mathrm{Cl}(S)) \subset s\mathrm{Cl}(S)$ for every subset S of X [11, Lemma 4.14]. Let $x \notin \mathrm{Int}(\mathrm{Cl}(U))$. Then $x \in X\text{-}\mathrm{Int}(\mathrm{Cl}(U)) \in \mathrm{SO}(X, \tau)$ and $U \cap X\text{-}\mathrm{Int}(\mathrm{Cl}(U))) = \emptyset$ since $U \in \mathrm{PO}(X, \tau)$. This shows that $x \notin s\mathrm{Cl}(U)$. Therefore, we obtain $s\mathrm{Cl}(U) = \mathrm{Int}(\mathrm{Cl}(U))$. The converse is obvious since $S \subset s\mathrm{Cl}(S)$ for every subset S of X.

LEMMA 3.2. *Let U be a subset of a space (X, τ). Then $U \in \tau^{\alpha}$ if and only if U is feebly open in (X, τ).*

PROOF. It is shown in Lemma 4.12 of [11] that $U \in \tau^{\alpha}$ if and only if there exists $G \in \tau$ such that $G \subset U \subset \mathrm{Int}(\mathrm{Cl}(G))$. Therefore, Lemma 3,1 completes the proof.

THEOREM 3.1. *A function $f : X \to Y$ is $a.\alpha.c.$ (resp. α-continuous) if and only if it is almost feebly continuous (resp. feebly continuous).*

PROOF. This is an immediate consequence of Lemma 3.2.

THEOREM 3.2 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*
(a) f is $a.\alpha.c.$
(b) For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau^{\alpha}$ containing

x such that $f(U)\subset Int(Cl(V))$.

(c) $f^{-1}(F)$ *is* α*-closed in* (X, τ) *for every regular closed set* F *of* (Y, σ).

PROOF. This is obvious.

The topology having the regular open sets in (X, τ) as a basis is called the *semi-regularization* of τ and is denoted by τ_s.

THEOREM 3.3 *For a function* $f:(X, \tau)\to(Y, \sigma)$, *the following are equivalent*:

(a) $f:(X, \tau)\to(Y, \sigma)$ *is a.α.c.*

(b) $f:(X, \tau)\to(Y, \sigma_S)$ *is* α*-continuous.*

(c) $f:(X, \tau^{\alpha})\to(Y, \sigma)$ *is almost-continuous.*

(d) $f:(X, \tau^{\alpha})\to(Y, \sigma_S)$ *is continuous.*

PROOF. Every $V\in\sigma_S$ is the union of regular open sets of (Y, σ). Therefore, (a) is equivalent to (b). It is obvious that (a), (c) and (d) are all equivalent.

4. Some properties

LEMMA 4.1. *Let* A *be a subset of a space* (X, τ). *If either* $A\in SO(X, \tau)$ *or* $A\in PO(X, \tau)$, *then* $A\cap V$ *is* α*-open in the subspace* $(A, \tau/A)$ *for every* $V\in\tau^{\alpha}$.

PROOF. This follows from [13, Lemma 2] and [8, Lemma 1.1].

It is shown in [6, Proposition 4] that $f:X\to Y$ is almost feebly continuous and A is open in X then the restriction $f|A:A\to Y$ is almost feebly continuous. As an improvement of this result, we have the following.

THEOREM 4.1. *Let* $f:(X, \tau)\to(Y, \sigma)$ *be an a.α.c. function. If either* $A\in SO(X, \tau)$ *or* $A\in PO(X, \tau)$, *then the restriction* $f|A:(A, \tau/A)\to(Y, \sigma)$ *is a.α.c.*

PROOF. Let V be a regular open set of (Y, σ). Since f is *a.α.c.*, $f^{-1}(V)\in\tau^{\alpha}$ and by Lemma 4.1 $f^{-1}(V)\cap A=(f|A)^{-1}(V)\in(\tau/A)^{\alpha}$. Therefore, $f|A$ is *a.α.c.*

In [3, Theorem 2.1] and [10, Corollary 4], it is shown that if $f:X\to Y$ is an almost-continuous injection and Y is Hausdorff then X is Hausdorff. The following theorem is a slight improvement of this result.

THEOREM 4.2. *If* $f:(X, \tau)\to(Y, \sigma)$ *is an a.α.c. injection and* (Y, σ) *is*

Hausdorff, then (X, τ) is Hausdorff.

PROOF. Since f is *a.α.c.*, by Theorem 3.3 $f:(X, \tau^{\alpha})\rightarrow(Y, \sigma)$ is almost-continuous. Since (Y, σ) is Hausdorff, so is (X, τ^{α}) [3, Theorem 2.1]. It follows from the proof of [11, Corollary 4.7] that (X, τ) is Hausdorff.

In [11, Theorem 4.9], the present author showed that if $f, g: X\rightarrow Y$ are α-continuous and Y is Hausdorff then $\{x\in X | f(x)=g(x)\}$ is α-closed in X. The following theorem shows that the assumption "α-continuous" in this result can be replaced by "*a.α.c.*"

THEOREM 4.3. *If $f, g:(X, \tau)\rightarrow(Y, \sigma)$ are a.α.c. and (Y, σ) is Hausdorff, then $\{x\in X | f(x)=g(x)\}$ is α-closed in (X, τ).*

PROOF. Since $f, g:(X, \tau)\rightarrow(Y, \sigma)$ are *a.α.c.*, by Theorem 3.3 $f, g:(X, \tau^{\alpha})\rightarrow(Y, \sigma_S)$ are continuous. Since (Y, σ_S) is Hausdorff, $\{x\in X | f(x)=g(x)\}$ is closed in (X, τ^{α}) and hence it is α-closed in (X, τ).

A function $f:(X, \tau)\rightarrow(Y, \sigma)$ is said to be *semi-weakly continuous* [12] if for each $x\in X$ and each $V\in\sigma$ containing $f(x)$, there exists $U\in \mathrm{SO}(X, \tau)$ containing x such that $f(U)\subset s\mathrm{Cl}(V)$.

LEMMA 4.2. *A function $f:(X, \tau)\rightarrow(Y, \sigma)$ is semi-weakly continuous if and only if $f^{-1}(V)\in SO(X, \tau)$ for every regular open set V of (Y, σ).*

PROOF. Necessity. Let V be a regular open ret of (Y, σ). For each $x\in f^{-1}(V)$, there exirts $U_X\in\mathrm{SO}(X, \tau)$ containing x such that $f(U_X)\subset s\mathrm{Cl}(V)$. By lemma 3.1, we have $s\mathrm{Cl}(V)=\mathrm{Int}(\mathrm{Cl}(V))=V$ and hence $x\in U_X\subset f^{-1}(V)$. Therefore, it follows from Theorem 2 of [4] that $f^{-1}(V)\in\mathrm{SO}(X, \tau)$.

Sufficiency. *Let* $x\in X$ and $f(x)\in V\in\sigma$. Put $U=f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$, then $x\in U\in \mathrm{SO}(X, \tau)$ and $f(U)\subset\mathrm{Int}(\mathrm{Cl}(V))=s\mathrm{Cl}(V)$ by Lemma 3.1. This shows that f is semi-weakly continuous.

REMARK 4.1. Semi-weak continuity and η-continuity are independent of each other. In Example 2.1, f is η-continuous but it is not semi-weakly continuous since $\{x\}$ is regular open in (Y, σ) and $\{a\}\notin\mathrm{SO}(X, \tau)$. Moreover, a semiweakly continuous function is not necessarily η-continuous as the following example shows.

EXAMPLE 4.1. Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma=\{\emptyset, \{a\}, \{b, c\}, X\}$. Then the identity function $f:(X, \tau)\to(X, \sigma)$ is semi-weakly continuous. However, f is not η-continuous since $\{b, c\}$ is regular open in (X, σ) and $f^{-1}(\{b, c\})$ does not satisfy (a) of Definition 2.4.

THEOREM 4.4. *A function $f:(X, \tau)\to(Y, \sigma)$ is a.α.c. if and only if and only if it is η-continuous and semi-weakly continuous.*

PROOF. Necessity. Suppose that f is *a.α.c.* By Theorem 2.1, f is η-continuous. Since $\tau^{\alpha}\subset SO(X, \tau)$, by Lemma 4.2 f is semi-weakly continuous.

Sufficiency. Let V be any regular open set of (Y, σ). Since f is semi-weakly continuous, by Lemma 4.2 $f^{-1}(V)\in SO(X, \tau)$. Moreover, f is η-continuous, $f^{-1}(V)\in PO(X, \tau)$ and hence $f^{-1}(V)\in\tau^{\alpha}$ [11, Lemma 3.1]. Therefore, f is *a.α.c.*

A function $f:(X, \tau)\to(Y, \sigma)$ is said to be *α-irresolute* [5] if $f^{-1}(V)\in\tau^{\alpha}$ for every $V\in\sigma^{\alpha}$. We obtain a characterization of α-irresolute functions by utilizing *a.α.c.* functions.

LEMMA 4.3. *Let A be a subset of a space (X, τ). Then $A\in\tau^{\alpha}$ if and only if there exists a regular open set O in (X, τ) and a nowhere dense set N such that $A=O-N$.*

PROOF. This follows easily from Proposition 4 of [9] and the proof.

THEOREM 4.5. *A function $f:(X, \tau)\to(Y, \sigma)$ is α-irresolute if and only if f is a.α.c. and $f^{-1}(N)$ is α-closed in (X, τ) for every nowhere dense set N in (Y, σ).*

PROOF. Necessity. Assume that f is α-irresolute. It is obvious that f is *a.α.c.* Let N be nowhere dense in (Y, σ). Then, $\mathrm{Int}(\mathrm{Cl}(N))=\emptyset$ and

$$Y-N\subset Y=Y-\mathrm{Int}(\mathrm{Cl}(N))=\mathrm{Cl}(\mathrm{Int}(Y-N)).$$

Therefore, we obtain $Y-N\subset\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(Y-N)))$. This shows that $Y-N\in\sigma^{\alpha}$. Therefore, $f^{-1}(Y-N)\in\tau^{\alpha}$ and $f^{-1}(N)$ is α-closed in (X, τ).

Sufficiency. Let $V\in\sigma^{\alpha}$. By Lemma 4.3, $V=O-N$, where O is regular open in

(Y, σ) and N is nowhere dense in (Y, σ). By the hypothesis, $f^{-1}(O) \in \tau^{\alpha}$ and $f^{-1}(N)$ is α-closed in (X, τ) and hence we have

$$f^{-1}(V) = f^{-1}(O) \cap (X - f^{-1}(N)) \in \tau^{\alpha}.$$

This shows that f is α-irresolute.

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