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# ATHWART IMMERSIONS WITH CODIMENSION $\mathrm{p}>2$ INTO EUCLIDEAN SPACE 

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#### Abstract

In this paper we define the athwart immersions with codimension $\mathrm{p}>2$ into Euclidean space. Some results supported by geometric examples have been established. A comparison study has been carried out throughout the paper.


## 1. Introduction

In this work, we are concerned with a problem, namely, the athwart immersions of $n$-manifolds into $(n+p)$-Euclidean space $E^{n+p}$ where $p>2$. In case of $p=1$, the problem has been considered by Robertson and Craveiro de Carvalho [4]. The athwartness definition as it has been given in [4] may be stated as follows: Let $M$ and $N$ be $C^{\infty}$, closed, connected, $n$-manifolds and let $f$ and $g$ be smooth immersions of $M$ and $N$, respectively, into $E^{n+1}$. We say that $f$ is athwart to $g$-written $f \cap g$-if and only if $f(M)$ and $g(N)$ have no tangent hyperplane in common.

In [4], the question: In what curcumistances is $f \cap g$ ? has been answered. In fact, the following theorems have been proved.

THEOREM (i). Let $f: M \longrightarrow E^{n+1}$ and $g: N \longrightarrow E^{n+1}$ be smooth immersions of the $n$-manifolds $M$ and $N$. If $f(M)$ has two tangent $n$-planes such that one meets $g(N)$ and the other does not, then $f$ is not athwart to $g$.

THEOREM (ii) Let $f: M \longrightarrow E^{n+1}$ and $g: N \longrightarrow E^{n+1}$ be immersions such that $f(M) \cap g(N) \neq \emptyset$. Then $f$ is not athwart to $g$.

THEOREM (iii). Let $f$ and $g$ be two immersions of the unit circle $S^{1}$ in $E^{2}$. If $f \not A g$, Then the image of one of the immersions is inside all the loops of the other.

THEOREM (iv). Let $f: M \longrightarrow E^{n+1}$ and $g: N \longrightarrow E^{n+1}$ be immersions such that $f \cap g$. Then one of the manifolds, say $M$, is diffeomorphic to the $n$-dimensional unit sphere $S^{n}, f$ is an imbedding with starshaped inside and $g(N)$ is
contained in the interior of the kernel of the inside of $f$.
In [1], the problem of athwart immersions with codimension 1 into hyperbolic space $H^{n+1}$ has been considered. The definition of athwartness in this case is given in terms of tangent totally geodesic hypersurfaces instead of tangent hyperplanes in case of the ambient Euclidean space. Theorems (i-iv) have been proved in [1] to be valid in case of $H^{n+1}$.

Recently, the problem of athwart immersions into sphere has been studied in [2]. The results obtained in [1] were found to be nonconsistant with those for the case of $E^{n+1}$ and $S^{n+1}$ as ambient spaces. One of the reasons of the nonconsistency is the existence of conjugate points on the sphere surface. When restricting oneself to athwart immersions in an open hemisphere instead of the whole of the sphere, the results were found to be completely consistent with those of the immersions in $E^{n+1}$ and $H^{n+1}$.

In both [1] and [2] the central projection map has been used successfully to construct a nice correspondence between athwart immersions in hyperbolic space as the open hemisphere and the same problem in the Euclidean space.

## 2. Definitions and backgrounds

We begin this article with mentioning what is meant by athwart immersions of smooth, closed, $n$-manifolds into $E^{n+p}$, where $p>2$ is an integer.

Let $M$ and $N$ be $C^{\infty}$, compact, $n$-manifolds and let $f: M \longrightarrow E^{n+p}$ and $g$ : $N \longrightarrow E^{n+p}(p \geqslant 2)$ be $C^{\infty}$ immersions of $M$ and $N$, respectively. The immersion $f$ is said to be athwart to $g$ (denoted by $f \cap g$ ) if there is no linear subvariety of dimension $n$ tangent to both $f(M)$ and $g(N)$. In this way of definition, the immersions $f$ and $g$ are not athwart if there exists a pair of points $p \in M$ and $q \in N$ such that $T_{p} M=T_{q} N$ where $T_{p} M$ denotes the tangent space of $M$ at $p$.

An immersion $f: M \longrightarrow E^{n+p}(p \geqslant 2)$ of an $n$-manifold $M$ is called substantial if $f(M)$ is not contained in an $(n+1)$-subariety of $E^{p+n}$. In other words a substantial immersion is a sizable immersion.

REMARK. If the two immersions $f$ and $g$ are non-substantial and contained together in the same $(n+1)$-subvariety of $E^{n+p}$, then all the results concerning athwart immersions (Theorems (i)-(iv)) are applied. This particular situation represents a reasonable motivation to restrict ourselves to dealing with substantial
immersions or non-substantial immersions which are not contained in the same linear subvariety.

Now, in the following simple geometric example we show that theorems (i) and (ii) are not valid for immersions of codimension $p>2$.

Example (i) Consider $M$ and $N$ to be $S^{1}$. Let $f$ and $g$ be two imbeddings of $S^{1}$ in $E^{3}$ such that $f$ imbedds $S^{1}$ as a unit circle in the $x y$-plane in $E^{3}$ with center at 0 and $g$ does the same thing in the $y z$-plane. Clearly, $f\left(S^{1}\right)$ has two tangent lines $T_{1}$ and $T_{2}$ where $T_{1}$ meets $g\left(S^{1}\right)$ and $T_{2}$ does not(See next Fig.). In the figure we could not find a common tangent line to both $f\left(S^{1}\right)$ and $g\left(S^{1}\right)$. Hence $f$ is athwart to $g$.


EXAMPLE (ii). If we consider the same figure, we should have two immersions $f$ and $g$ such that $f\left(S^{1}\right) \cap g\left(S^{1}\right) \neq 0$ while $f$ is athwart to $g$.

As the inside (or outside) of a loop in $E^{3}$ as well as the kernel of the immersion in case of codimension greater than one do not make sense we could not give examples to show to what extent theorems (iii) and (iv) are valid.

## 3. Main Work

The main work of this paper which is precisely incorporated in finding some necessary and sufficient conditions for immersions to be athwart lies in three different parts (a), (b), and (c). In part (a), we investigate the case when the two immersions are substantial. The second one, part (b), deals with the case when one of the considered immersions is substantial and the other is not. The case of two non-substantial immersions is studied in part (c).

From now on, $M$ and $N$ are taken to be closed $C^{\infty}, n$-manifolds and all immersions are smooth enough for all discussions to make sense. The integer $p$ always satisfies $p>2$.

## Part (a): Wholly Substantial Case.

PROFOSITION (i) Let $f: M \longrightarrow E^{n+1}$ and $g: N \longrightarrow E^{n+p}$ be two smooth immersions such that $f(M)$ is contained on the surface of a convex closed hypersurface $\tilde{M}$ and $g(N)$ is contained in the open interior of $\tilde{M}$, then $f$ is athwart to $g$.

PROPOSITION (ii). Let $f$ and $g$ (in prop. (i)) be both contained on a surface of a strictly convex closed hypersurface $\tilde{N}$ such that $f \cap g=0$, then $f$ is atwart to $g$.

PROOF OF PROPOSITION (i) In contrary to the required result, assume that $f$ is not atwart to $g$. Then there exists a common $n$-linear subvariety $H^{n}$ of $E^{n+p}$ tangent to both $f(M)$ and $g(N)$. Let $p_{1} \in f(M)$ and $p_{2} \in g(N)$ represent the points of tangency. The tangency of $H^{n}$ to both $f(M)$ and $g(N)$ ensures that there exists a tangent hyperplane $H^{n+p-1}$ to the convex hypersuface $\tilde{M}$ at $p_{1}$ which contains $H^{n}$. In this way, the tangent hyperplane $H^{n+p-1}$ divides $\tilde{M}$ into two (or more) pieces contradicting the convexity of $\tilde{M}$. The contradiction shows that $f$ is athwart to $g$.

ROOF OF PROPOSITION (ii). Let $f$ and $g$ be two immersions contained on the surface of a strictly convex hypersurface $\tilde{N}$ of $E^{n+p}$ and $f(M) \cap g(N)=0$ as being mentioned in the proposition. Let us carry out the proof by the way of contradiction as it has just been done with prop. (i).

Assume that $f$ and $g$ are non-athwart, consequently there exists an $n$-linear subvariety $H^{n}$ which represents a tangent space to both $f(M)$ and $g(N)$ at $p_{1}$ and $p_{2}$, respectively. Hence, there would be a straight line, say $L$, which touches $\widetilde{N}$ at two different points. Consider the tangent hyperplane $\widetilde{N}_{p_{1}}$ of $\widetilde{N}$ at $p_{1}$ which contains $L, p_{1}$ and $p_{2}$. This hyperplane, in view of the above arguments, touches $\widetilde{N}$ at $p_{1}$ and intersects $\widetilde{N}$ again at $p_{2}$ contradicting the fact that $\tilde{N}$ is strictly convex and the proof is now complete.

REMARKS. (a) In the same way of proof we can easily show that if the condition $f \cap g=\emptyset$ in prop. (ii) is replaced by the term " $f$ intersects $g$ transver-
sally", the proposition is still true.
(b) Athwartness property of submanifolds can be adopted as a good measure of strict convexity in the following sense: If each two transversally intersecting submanifolds on the surface of a closed hypersurface $\tilde{N}$ of $E^{n+p}$ are athwart, then $\widetilde{N}$ is strictly convex. The proof of this idea is now clear in the light of the above mentioned propositions.

Part (b): Semi-substantial Case.
PROPOSITION (iii). If $g(N)$ is contained in an $(n+1)$-linear subvariety $H^{n+1}$ and $f(M)$ is substantial, then $f$ is athwart to $g$ in the following cases:
(a) $f(M) \cap H^{n+1}=0$
(b) $f(M)$ intersects $H^{n+1}$ transversally.
(c) $f(M)$ intersects $H^{n+1}$ tangentially but not along any tangent space of $g(N)$.

PROOF. (a) As $g(N)$ is contained wholly in $H^{n+1}$, then all tangent spaces of $g(N)$ are also contained in $H^{n+1}$. Hence, if $f(M) \cap H^{n+1}=\emptyset$, no tangent space of $f(M)$ is contained in $H^{n+1}$ and consequently, no tangent space of $g(N)$ coincides with a tangent space of $f(M)$.
(b) When $f(M)$ intersects $H^{n+1}$ transversally, there is no tangent space of $f(M)$ lying in $H^{n+1}$ and hence there is no common tangent $n$-linear subvariety between $f(M)$ and $g(N)$.
(c) We leave this part to the reader as a geometric exercise.

PROPOSITION (iv). If $g(N)$ is contained in an ( $n+1$-linear subvariety $H^{n+1}$ such that $\bigcup_{x \in N} T_{X} N \neq H^{n+1}$ and $f(M)$ intersects $H^{n+1}$ in the interior region of the kernel of the inside of $g(N)$, then $f \cap g$.

PROOF. The proof of this proposition depends heavily on a theorem established by B. Halpern [3] which states that; For a smooth immersion $g: N \longrightarrow$ $E^{n+1}, \operatorname{dim} N=n$, if $\bigcup_{x \in N} T_{X} N \neq E^{n+1}$ then $N$ is diffeomorphic to $S^{n}, g$ is an imbedding, the inside of $g(N)$ is starshaped and $E^{n+1}-\bigcup_{x \in N} T_{X} N$ is the interior of the kernel of the inside of $g$.

Taking this theorem into account and due to the fact that all tangent spaces of $g(N)$ are in $H^{n+1}$, and moreover, that $f(M)$ intersects $H^{n+1}$ in the interior of the kernel of the inside of $g(N)$, there would be no tangent space of $f(M)$
coincident with a tangent space of $g(N)$.
It is worth mentioning that even if $f(M)$ is intersecting $H^{n+1}$ tangentially the proposition is still valid.

## Part (c): Wholly Non-substantial Case

The simplest case considered in this work is the one under consideration. The situation is now divided into two different portions, parallel and transversal cases. Proposition (v) deals with the first case while proposition (vi) treats the other one.

PROPOSITION (v). If the immersions $f$ and $g$ are non-substantial such that $f(M) \subset H_{1}^{n+1}$ and $g(N) \subset H_{2}^{n+1}$ and $H_{1}^{n+1} \cap H_{2}^{n+1}=\emptyset$, then $f \not A g$.
(The proof of this proposition is too simple).
PROPOSITION (vi). For the immersions $f$ and $g$ in proposition (v), if $H_{1}^{n+1}$ $\cap H_{2}^{n+1}=\emptyset$, then $f$ is athwart to $g$ if and only if $H_{1}^{n+1} H_{2}^{n+1}$ does not represent a common tangent space to both $f(M)$ and $g(N)$.

The proof of this proposition comes out as a direct geometric result. As an application, the following geometric attitudes represent some athwartness cases:
(a) $f(M) \cap H_{1}^{n+1} \cap H_{2}^{n+1}=0$
(b) $f(M)$ intersects $H_{1}^{n+1} \cap H_{2}^{n+1}$ transversally.

Same things are true when replacting $f(M)$ by $(N)$.
NOTE. All the results of this paper can be proved true when considering immersions into hyperbolic spaces instead of the Euclidean ones. One can use. to acheive this aim-the central projection map as it has been done in [1]. Considering immersions into elliptic space needs careful study as we expect a great alterations in the results.

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