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A RECURSIVE ALGORITHM TO INVERT MULTIBLOCK CIRCULANT MATRICES

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Abstract: Circulant and multiblock circulant matrices have many important applications, and therefore their inverses are of considerable interest. A simple recursive algorithm is presented to compute the inverse of a multiblock circulant matrix. The algorithm only uses complex variables, roots of unity and normal matrix/vector operations.

1. Introduction

Circulant matrices have applications in physics, Fourier analysis, geometry, probability, statistics and other areas of mathematics. Davis [2] devotes an entire chapter to the analysis and determination of circulant inverses and sets of equations. John [3] describes the application of block and multiblock circulants to cyclic designs in Factorial experiments. The book by John and Quenoville [4] contains background material on experimental design.

A matrix is a circulant if the i-th row may be obtained from the (i-1)th row by a right circular shift of one element. For example the following matrix is a 3×3 circulant:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

A block circulant matrix is a partitioned matrix with blocks patterned in the circulant fashion, and where each of the blocks itself is a circulant. For example the following matrix is a block circulant, it is comprised of 2×2 circulant blocks.

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$$

Notice that a block circulant matrix need not be a circulant matrix.

An M-block circulant matrix is a partitioned matrix with blocks patterned in the circulant fashion, and where each of the blocks is an (M-1)-block circulant. A 0-block circulant is a circulant, and a 1-block circulant is a block circulant.

Because of the special structure of a circulant, the matrix is completely specified by its first row. Similarly the first row of an M-block circulant plus the sizes of the partitions of the subblocks completely determine the M-block circulant. The first row of a circulant or multiblock circulant may be represented by a vector. In the remainder of this paper, we will regularly use the vector representation in place of the matrix representation.

Given a circulant, block circulant or multiblock circulant matrix A, there are two problems of interest: 1) determine A^{-1} when it exists, and 2) solve the set of linear equations $Ax=b$. Because of the special structure of multiblock circulants, these two problems are essentially equivelent. Davis [2] has shown that if an M-block circulant is invertible, then its inverse is also an M-block circulant with the same structure. Therefore the problems mentioned above may be summarized as follows: given a vector representing the first row of an M-block circulant A, and the sizes of the various subblocks, compute, if possible, a new vector which is the first row of the M-block circulant inverse of A. We will present an algorithm which determines the first row of the inverse of an M-block circulant is less time than the Gaussian elimination method used for solving the corresponding system of equations.

We will show that the number of multiplications to determine the first row of the inverse matrix is proportioned to the size of the matrix squared; this is a vast improvement over the classical methods.

2. Solution procedure

We will first review the procedure for inverting a circulant matarix. If a_1, a_2, $\cdots$, a_n are the elements of the first row of a circulant, then define α_1, α_2, $\cdots$, α_N as follows:

$$\alpha_k=\sum_{j=1}^{N}(r_k)^{(j-1)}a_j \tag{1}$$

where $r_k=\cos(2\pi(k-1)/N)+i\sin(2\pi(k-1)/N)$. Here i denotes the imaginary square root of minus one. The r_k's are the Nth roots of unity.

The circulant has an inverse if and only if $\alpha_k\neq0$ for $k=1, 2, \cdots, N$. In this case let $\beta_k=1/\alpha_k$ and define b_1, b_2, $\cdots$, b_N as follows:

$$b_j = \frac{1}{N} \sum_{k=1}^{N} (\bar{r}_k)^{(j-1)} \beta_k \tag{2}$$

where $\bar{r}_k$ is the complex conjugate of r_k. Then $b_1, b_2, \cdots, b_N$ is the first row of the inverse circulant.

Next we consider the multiblock circulant case. Suppose we have a routine that will invert an (M-1)-block circulant. Then given $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_N$ vectors representing the first row of an M-block circulant, we perform the following steps. First, since each vector $\underline{a}_j$ is the first row of an $(M-1)$-circulant, the vectors given as follows $\underline{\alpha}_1, \underline{\alpha}_2, \cdots, \underline{\alpha}_N$

$$\underline{\alpha}_k = \sum_{j=1}^{N} (r_k)^{(j-1)} \underline{a}_j \tag{3}$$

are also the first row of (M-1)-block circulants.

This is true because a linear combination of multiblock circulants is a multiblock circulant, and equation (3) is a linear combination of the $\underline{a}_j$'s. Second we invoke our subroutine to invert an (M-1)-block circulant. We will call our subroutine N times, for each α_k, $k=1, 2, \cdots, N$. Let the resultant vectors, which represent the first rows of the corresponding inverses be $\beta_1, \beta_2, \cdots, \beta_N$. Third, the vectors $\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_N$ which are the first row of the inverse of the original M-block circulant are computed as follows:

$$\underline{b}_j = \frac{1}{N} \sum_{k=1}^{N} (\bar{r}_k)^{(j-1)} \beta_k \tag{4}$$

Notice that the procedure used to invert a circulant (equations 1 and 2 and scalar inversions) is essentially the same as that for inverting an M-block circulant (equations 3 and 4 and call to an inversion routine). The form of the equations is the same, the difference is that for circulatns we have scalars, while for M-block circulants we have vectors. Recursion allows us to combine the cases and develop a simple algorithm for the finding the inverse of an M-block circulant.

We will now describe a routine that accepts N vectors $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_N$ where each vector is of length P. Our routine will produce a new sequence of vectors $\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_N$, each of these vectors is also a length P. The procedure to determine the $\underline{b}_i$'s is a combination of the methods discussed previously. The case $P=1$, where each $\underline{a}_i$ is a scalar, is permitted and is in fact necessary.

Given an M-block circulant, we define an array S, were S(i) is the number of sub-blocks of the i-th block circulant. For example if we have 3-block circulant then we would assign to S(3) the number of 2-block circulants. To

S(2) we would assign the number of 1-block circulants. S(1) would contain the number of 0-block circulants (remember these are ordinary circulants). And finally S(0) would contain the length (or size) of the 0-block circulants.

For an M-block circulant, the total length T of the first row is given by:

$$T=S(0)^{*}S(1)^{*}\cdots^{*}S(M).$$

The N and P that we used previously for the number of a vectors and the length of the a vectors may be computed using the following two equations:

$$N=S(M)$$

$$P=T/N$$

Given a main routine which assigns values to the A vector (this is the first row of the M-block circulant being inverted), the S vector and M variable, then a typical call to the subroutine would be of the form:

CALL BLKCIRC(S, M, A, B, FLAG)

After returning from the subroutine, the B vector would contain the first row of the inverse matrix whenever the inverse exists which is specified by FLAG=0.

The structure of the BLKCIRC routine is as follows:

```
BLKCIRC: (S, M, A, B, FLAG)
  FLAG=0
  N=S(M)
  If M=0 then P=1 ELSE P=S(0)*S(1)*...*S(M-1)
  FILL THE N ALPHA VECTORS USING EQUATION 3 (or 1)
              THESE VECTORS ARE OF LENGTH P.
  IF P=1 THEN DO
    FOR I=1 TO S(0)
      IF ALPHA=0 THEN DO
        FLAG=1
        PETURN
      ELSE
        BETA=1/ALPHA
      END IF
    END FOR
  ELSE (When P>1)
      For I=1 TO N
        CALL BLKCIRC(S, M-1, ALPHA, BETA, FLAG)
        IF FLAG=1 THEN RETURN
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  END FOR
END IF
NOW FILL THE B VECTORS USING EQUATION 4 (or 2)
RETURN
  END; BLKCIRC
```

3. Analysis of the solution procedure

To examine the efficiency of the algorithm decribed above to determine the inverse of an M-block circulant, we will use the notation of the previous section and only count multiplications. We ignore the divisions because they are of linear order of the size of the matrix, and as we shall see multiplications are of quadratic order.

Given an M-block circulant matrix where S_i is the number of sub-blocks in the i-th block circulant and T is the size of the matrix. Then $T = S_0 * S_1 * S_2 * \cdots * S_M$.

Let $F(k)$ equal the number of multiplications that are performed in the call to BLKCIRC that processes the k-th block circulant, i.e. BLKCIRC(S, K. A. B. FLAG).

There are three steps in this call to BLKCIRC

1) determine S_k vectors of size $S_0 * S_1 * \cdots S_{k-1}$ (the vectors α_i)
2) call BLKCIRC (S, K-1, A, B, FLAG) S_k times
3) determine S_k vectors of size $S_0 * S_1 * \cdots * S_{k-1}$ (the vectors $\underline{b}_j$).

For $k=0$, the case of a circulant matrix, step one is carried out by equation (1) and step three is carried out by equation (2) Step two corresponds to $\beta_k = 1/\alpha_k$.

We assume the powers of the roots of unity are availabe in an array and hence no multiplications are required to obtain these, we simply use a table look up.

For each k in equation (1), $1 \leq k \leq n$, n multiplications are required. Similarly, for each j, $1 \leq j \leq n$, in equation (2), n multiplications are required. Hence $F(0) = 2n^2$ or $F(0) = 2S_0^2$

For $M \geq 1$ steps two and three are carried out by equations (3) and (4) respectively.

For step one each $\underline{a}_j$ in equation (3) is a vector of size $S_0 * S_1 * \cdots * S_{M-1}$ and $N = S_M$. Hence for each k, $1 \leq k \leq M$ there are $(S_0 * S_1 * \cdots * S_{M-1}) * S_M^2$ multiplications.

Similarly, step three requires the same numbers of operations. Hence.

$$F(M)=2S_0{}^{*}S_1{}^{*}\cdots{}^{*}S_{M-1}{}^{*}S_M^2+S_M F(M-1)$$

and by induction

$$F(M)=2S_0{}^{*}S_1{}^{*}\cdots{}^{*}S_M(S_0+S_1+\cdots+S_M).$$

It is reasonable to assume that $S_i>1$ for each i and then it can eaily be shown that

$$F(M)<2(S_0{}^{*}S_1{}^{*}\cdots{}^{*}S_M)^2$$

or

$$F(M)<2T^2$$

where T is the size of the multiblock circulant.

With T^2 additional multiplication, we can compute $A^{-1}y$ and hence solve the system $A_x=y$.

We have ignored the overhead of the recursive calls to BLKCIRC.

We note further that the numerical problem of what is zero, and any concern with the condition number of the matrix have not been addressed. These problems should be a fruitful area for further study.

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