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TRANSNORMAL SYSTEMS ON THE CAYLEY PROJECTIVE PLANE

by Kwang Sung Park

1. Introduction.

Transnormal systems in complete connected Riemannian manifolds were introduced by J. Bolton([1], [2]). He were concentrated on those systems containing hypersurfaces. As a special case, in spheres, his notion is equivalent to isoparametric families. Transnormal systems containing hypersurfaces in complex projective spaces and quaternionic projective spaces were studied in [6]. The method used in [6] is based on the observation that those projective spaces are images of Riemannian submersions of spheres. But the Cayley projective plane KP^2 can not be the image of a Riemannian submersion of a sphere. And this is the only compact, simply connected symmetric space of rank one which is not included in [6]. Sometimes, topological structures of manifolds determine some of their geometric features. In this paper, we investigate homology groups of hypersurfaces in transnormal systems in KP^2 to obtain some restrictions on codimensions of singular foils.

2. Preliminaries.

A *system* on a connected manifold N is a partition of N into nonempty connected submanifolds called *foils*. A *transnormal system* on N is a system in which N is a complete connected Riemannian manifold such that any geodesic of N cuts the foils of the system orthogonally at none or all of its points. It is ***non-singular*** if all of its foils have the same dimension. Otherwise it is *singular*. J. Bolton proved the following results([1], [2]).

PROPOSITION 2.1. *Let $\mathscr{L}$ be a transnormal system in N with a foil of codimension* 1. *Then $\mathscr{L}$ has at most two singular foils. Furthermore.*

(i) If $\mathscr{L}$ has no singular foils, then it is a metric foliation.

(ii) If $\mathscr{L}$ has only one singular foil A, then either N has a vector bundle structure over A or N has double cover.

(iii) If $\mathscr{L}$ has two singular foils, then N has the topological structure of two sphere bundles over singular foils, glued together along their boundaries.

In the case(iii), let F be the homotopy fiber of the inclusion $j: M \longrightarrow N$ and let A_1, A_2 be the two singular foils of codimensions k_1, k_2. Let ϕ_i: $M \longrightarrow A_i$ the canonical map. Then the homology of the homotopy fiber F is given as follows ([4]):

(k, k_2)	$(H_i(F;Z))$	
$k_1 \neq k_2$ no twists	Z	$i=0$ or $i \equiv k_1, l \bmod (k_1+k_2)$
	$Z \oplus Z$	$i>0$ and $i \equiv 0 \bmod (k_1+k_2)$
$k_1 = k_2$ no twists	Z	$i=0$
	$Z \oplus Z$	$i>0$ and $i \equiv 0 \bmod (k_1)$
$k_1 > k_2 = 1$ ϕ_1 twisted	Z	$i=0$ or $i \equiv \pm 1 \bmod (2k_1+2)$
	$Z \oplus Z$	$i>0$ and $i \equiv 0 \bmod (2k_1+2)$
	Z_2	$i \equiv k_1, k_1+1 \bmod (2k_1+2)$
$k_1 = k_2 = 1$ ϕ_1 twisted ϕ_2 not twisted	Z	$i=0$ or $i \equiv 3 \bmod (4)$
	$Z \oplus Z_2$	$i \equiv 1 \bmod (4)$
	Z_2	$i \equiv 2 \bmod (4)$
	$Z \oplus Z$	$i>0$ and $i \equiv 0 \bmod (4)$
$k_1 = k_2 = 1$ ϕ_1, ϕ_2 both twisted	Z	$i=0$
	$Z \oplus Z$	$i>0$ and $i \equiv 0 \bmod (3)$
	$Z_2 \oplus Z_2$	$i \equiv 1 \bmod (3)$

Table 1

W. Thurston proved the following([8])

PROPOSITION 2.2. *A closed manifold N has a codimension one foliation if and only if its Euler characteristic is zero.*

3. Results.

Let $\mathscr{L}$ be a transnormal system on the Cayley projective plane KP^2 with a hypersurface M. Then it has at most two singular foils. If it has no singular foils, then it gives a codimension one foliation. Thus the Euler characteristic of KP^2 should be zero. But the homology group of KP^2 is given as follows:

$$H_q(KP^2) = \begin{cases} Z \text{ for } q=0,\ 8,\ 16 \\ 0 \text{ otherwise} \end{cases}$$

Thus the Euler characteristic of KP^2 is 3, a contradiction. Suppose that $\mathscr{L}$ has only one singular foil. Note that KP^2 is compact, and hence KP^2 can not have a vector bundle structure. If KP^2 has a double cover $\tilde{N}$, then we have the following Gysin exact sequence

$$\cdots \longrightarrow H^{q-1}(KP^2) \longrightarrow H^q(KP^2) \longrightarrow H^q(\tilde{N}) \longrightarrow H^q(KP^2) \longrightarrow \cdots$$

with Z_2 coefficients. When $q=0$, we have the following exact sequence

$$0 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow 0,$$

a contradiction. Thus the case(iii) in Proposition 2.1 can not occur, and hence we have the following

PROPOSITION 3.1. Let $\mathscr{L}$ be a transnormal system on KP^2. Then it has two singular foils.

Let A_1 and A_2 be the two singular foils of $\mathscr{L}$ with codimensions k_1 and k_2, respectively. Then we have the following decomposition

$$KP_2 = DA_1 \cup_M DA_2,$$

where DA_1 and DA_2 are k_1- and k_2- sphere bundles over A_1 and A_2, respectively. Then $H_*(F)$ $H_*(KP^2)$ is a term of a spectral sequence which converges to $H_*(M)$. Thus we can compute the homology of M by Table 1. Note that dim $M=15$ and hence $H_q(M)=H_{15-q}(M)$ by the Poincare duality. Using this fact, we can exclude almost all of the possibilities of (k_1, k_2). For example, if $k_1 \neq k_2$ and $k_1, k_2 > 7$, then we have $H_7(M)=0$ and $H_8(M)=Z$, a contradiction. In fact, we have

PROPOSITION 3.2. *Let $\mathscr{L}$ be a transnormal system on KP^2 and let k_1, k_2 be the two codimensions of singular foils of $\mathscr{L}$. Then (k_1, k_2) is eqval to* (1, 1), (7, 4) *or* (15, 7).

The case (15, 7) exists. In fact, A_1 is a point and A_2 is an 8-sphere. But we don't know if the other cases exist.

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