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ASCOLI'S THEOREM AND THE PURE STATES OF A C*-ALGEBRA

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Abstract: A version of Ascoli's Theorem (equating compact and equicontinuous sets) is presented in the context of convergence spaces. This theorem and another, (involving equicontinuity) are applied to characterize compact subsets of quasi-multipliers of a C^*-algebra B, and to characterize the compact subsets of the state space of B.

The classical Ascoli Theorem states that, for pointwise pre-compact families F of continuous functions from a locally compact space Y to a complete Hausdorff uniform space Z, equicontinuity of F is equivalent to relative compactness in the compact-open topology([4] 7.17). Though this is one of the most important theorems of modern analysis, there are some applications of the ideas inherent in this theorem which are not readily accessible by direct appeal to the theorem. When one passes to so-called "non-commutative analysis", analysis of non-commutative C^*-algebras, the analogue of Y may not be relatively compact, while the conclusion of Ascoli's Theorem still holds. Consequently it seems plausible to(1) establish a more general Ascoli Theorem which will directly apply to these examples. (2)

1. PT-Classes

Many theorems of a topological nature in analysis are more readily stated in terms of convergent sequences or nets, rather than in terms of the more "basic" notions of open and closed sets. In fact, certain types of convergence in some important thorems (such as "almost everywhere convergence") may not even be convergence relative to some topology. This has prompted the study of "convergence spaces" (see [1], [2], and [3] for instance). However the axiomatic treatment of "convergence spaces" seems to have been for the most part in terms of filters instead of nets. The notion of a "convergence space" is just as simple as that of a topological space, it is necessary for statement of

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(2) To understand the examples of this paper it will help it the reader has had some experience with the fundamental parts of the theories of locally convex spaces and C^*-algebras as in [7] and [8].

the general Ascoli Theorem, and so we shall set down a formulation of the axioms and basic facts concerning "convergent spaces" in terms of nets.

Let S be a set and $\mathscr{N}$ the class of all nets in S. For $s\in S$, the singleton $\{s\}$ has only one binary relation relative to which $\{s\}$ is a directed set—we write $\bar{s}$ for the net sending $s\in\{s\}$ to s. For nets $x|D\to S$ and $y|E\to S$ we write $x\vee y$ for the net whose domain is the product directed set $N\times D\times E$ and which sends $(n,d,e)\in N\times D\times E$ to x_d or y_e depending as n is odd or even.

A subclass $\mathscr{S}$ of $\mathscr{N}$ will be called a *PT-class* (*pseudo-topological class*) if the following axioms hold:

(1) $\bar{s}\in\mathscr{S}$ for all $s\in S$;

(2) $x\vee y\in\mathscr{S}$ whenever $x,y\in\mathscr{S}$ have a common subnet;

(3) $x\in\mathscr{N}$ is a member of $\mathscr{S}$ if and only if each subnet of x has a subnet belonging to $\mathscr{S}$.

It is a direct consequence of (3) that

(4) a subnet of a member of $\mathscr{S}$ also belongs to $\mathscr{S}$.

We say that $\mathscr{S}$ is Hausdorff if, for all $a,b\in S$,

(5) $a=b$ whenever $a\vee b\in\mathscr{S}$.

Let $\mathscr{S}$ be a PT-class. If each net in S has a subnet belonging to $\mathscr{S}$ we say that $\mathscr{S}$ is *precompact*. Let A be a subset of S. Then the class $\mathscr{S}(A)$ of all nets in A belonging to $\mathscr{S}$ is evidently a PT-class for A, and is Hausdorff if $\mathscr{S}$ is. We shall say that A is *precompact* if $\mathscr{S}(A)$ is. We say that A is *dense in* S if to each $x\in\mathscr{S}$ there exists $y\in\mathscr{S}(A)$ such that $x\vee y\in\mathscr{S}$.

If $[x\vee\bar{s}]$ belongs to $\mathscr{S}$ for $x\in\mathscr{S}$ and $s\in S$, we say that x *converges* to s (or s is a *limit* of x) and write $x\xrightarrow{\mathscr{S}}s$. If each $x\in\mathscr{S}$ has a limit, we say that $\mathscr{S}$ is *complete*. If $\mathscr{S}(A)$ is complete for $A\subset S$, we say that A is *complete*. A complete precompact set is said to be *compact*. Thus a compact set is one in which each net has a subnet with a limit.

New let S and T have PT-classes $\mathscr{S}$ and $\mathscr{T}$ respectively. Then a function $f|S\to T$ is said to be *continuous* if $f\circ x\in\mathscr{T}$ whenever $x\in\mathscr{S}$. It is elementary to check that precompact sets and compact sets are preserved by continuous functions.

A PT-class $\mathscr{S}$ will be called a *T-class* if it is closed under the formation of "upper diagonal nets". More precisely, let $m|D\to S$ be a net belonging to S and suppose that, for each $d\in D$, $s^{(d)}|E(d)\to S$ is a net such that $s^{(d)}\vee m_d\in\mathscr{S}$(i.e., $s^{(d)}\xrightarrow{\mathscr{S}}m_d$). Let $H\equiv D(\underset{d\in D}{X}E(d))$ be the product directed set and define $s|H\to s$ by letting $s_{(d,\gamma)}\equiv s^{(d)}_{\gamma(d)}$ for all $(d,\gamma)\in H$. If $s\vee m$ belongs to $\mathscr{S}$ for each such construction, then a PT-class $\mathscr{S}$ is called a *T-class.*

Let $\mathscr{S}$ be a T-class. Then, for $A\subset S$, it is evident that $\mathscr{S}(A)$ is a T-class as well.

EXAMPLE 1.1. topological space. Let S be a set with a topology σ and let $\mathscr{S}$ be the class of all nets which converge in S relative to σ. We shall say that $\mathscr{S}$ is the T-class induced by σ. That $\mathscr{S}$ satisfies the axioms of a PT-class is elementary. That $\mathscr{S}$ actually is a T-class follows from [4] 2.9. The notions of density, compactness, limit, and continuity are the same as the corresponding topological notions. Note however that, for $A\subset S$, $\mathscr{S}(A)$ is the PT-class induced by the relative topology on A only if A is closed. It is true that a subset A of S is compact relative to σ if and only if it is compact relative to $\mathscr{S}(A)$.

EXAMPLE 1.2. uniform space. Let S be a uniform space with uniformity Ω. Then the *T-class induced by* Ω is the class $\mathscr{S}$ of all Cauchy nets in S(3). For $A\subset S$ it is evident that $\mathscr{S}(A)$ is the T-class induced by the relativization of Ω to A. The notions of precompactness and completeness are the same relative to $\mathscr{S}$ as to Ω.

EXAMPLE. 1.3. continuous convergence. Let Y and Z be sets with PT-structures $\mathscr{G}$ and $\mathcal{Z}$ respectively. Let F be a set and $\phi|F\times Y\to Z$ a function. For $a\in F$ and $b\in Y$ we define the functions $a\phi$ and ϕb by

$$a\phi|Y\ni y\to a\phi y\equiv\phi(a,y),\quad \phi b|F\ni x\to x\phi b\equiv\phi(x,b).$$

We shall say that ϕ is a *dual mapping* if $a\phi$ is continuous for all $a\in Y$. Suppose ϕ is such. The continuous convergence PT-class $\mathscr{C}_\phi(F)$ consists of all nets x in F such that $x\phi y\in\mathcal{Z}$ for every $y\in\mathscr{G}$. For $A\subset F$ we write $\mathscr{C}_\phi(A)$ for $(\mathscr{C}_\phi(F))\,(A)$.

2. First Ascoil Theorem

Let $\phi, F, Y, Z, \mathscr{C}_\phi(F), \mathscr{G}$, and $\mathcal{Z}$ be as in Example 1.3 of the preceding section. Let in addition Z be a uniform space, and suppose that $\mathcal{Z}$ is the T-class induced by the uniformity on Z. We say that $A\subset F$ is an *E-set* if, for each $y|D\to Y$ belonging to $\mathscr{G}$ and each U in the uniformity on Z, there exists $c\in D$ such that $(a\phi y_d,\ a\phi y_e)\in U$ for all $a\in A$ and all $d,e\geqq c$. It is elementary to check that, when $\mathscr{G}$ is the T-class induced by some topology on Y, then $A\subset F$ is an E-set if and only it $\{a\phi : a\in A\}$ is equicontinuous. For $A\subset F$, we write $A\phi$ for the

(3) Since a uniform space S has a completion $\mathscr{C}$, it is easily seen that $S=\mathscr{C}(S)$ where $\mathscr{C}$ is the T-class an $\mathscr{C}$ induced by the topology and $\mathscr{C}$. Thus $\mathscr{S}$ actually is a T-class.

family $\{a\phi : a \in A\}$.

THEOREM 2.1. *Let $A \subset F$ be such that $A\phi y$ is precompact for all $y \in Y$. Then A is $\mathscr{G}_\phi(A)$-precompact if and only if A is an E-set.*

PROOF. Suppose first that A is an E-set, and let x be a net in A. Since $A\phi y$ is precompact for each $y \in Y$, it follows from Tychonov's Theorem that there is a subnet $\alpha | D \to A$ of x such that $\alpha\phi y$ is Cauchy in Z for each $y \in Y$. Let $y | E \to Y$ be a net belonging to Y, p be a continuous pseudo-metric on Z, and $\varepsilon > 0$. Since A is an E-set, there exists $s \in E$ such that

(6) $p(a\phi y_t,\ a\phi y_s) < \varepsilon/2$ for all $a \in A$ and $t \geqq s$.

Choose $c \in D$ such that $p(\alpha_d\phi y_s,\ \alpha_c\phi y_s) < \varepsilon/2$ for all $d \geqq c$; thus (6) yields, for $t \geqq s$ and $d \geqq c$,

$$p(\alpha_d\phi y_t,\ \alpha_c\phi y_s) \leqq p(\alpha_d\phi y_t,\ \alpha_d\phi y_s) + p(\alpha_d\phi y_s,\ \alpha_c\phi y_s) < \varepsilon.$$

Hence $\alpha\phi y$ is Cauchy in Z, and so a member of $\mathscr{Z}$. Thus A is $\mathscr{G}_\phi(A)$-precompact.

Now suppose that A is $\mathscr{G}_\phi(A)$-precompact, and assume that A is not an E-set. Then there exist a continuous pseudo-metric p on Z and a net $y | E \to Y$ belonging to $\mathscr{G}$ such that to each $s \in E$ corresponds some $m(s) \geqq s$ and $\alpha_s \in A$ such that

$$(7) \qquad p(\alpha_s\phi y_s,\ \alpha_s\phi y_{m(s)}) > \varepsilon.$$

Since A is $\mathscr{G}_\phi(A)$-precompact, the net $\alpha | E \to A$ defined by (7) has a subnet $x \in \mathscr{G}_\phi(A)$. Thus $x\phi y$ is Cauchy, which violates (7). Hence A is an E-set. Q.E.D.

EXAMPLE 2.2. classical Ascoli Theorem for locally compact spaces. Let Y be a locally compact topological space and $\mathscr{G}$ the induced T-class. Let Z be a complete uniform space, and let F be the family of all continuous Z-valued functions on Y. Let $f\phi y \equiv f(y)$ for all $f \in F$ and $y \in Y$. It is elementary to check that $\mathscr{G}_\phi(F)$ is the T-class induced by the compact-open topology on F; and that, for $A \subset F$, $\mathscr{G}_\phi(A)$ is precompact if and only if A is relatively compact relative to the compact-open topology. Thus Theorem 1 yields the following classical theorem.

COROLLARY 2.3. *Let $A \subset F$ be such that $\{f(y) : f \in A\}$ is precompact for all $y \in Y$. Then A is relatively compact (with respect to the compact-open topology) if and only if A is equicontinuous*

EXAMPLE 2.4. Let B be a C^*-algebra, which we may (and shall) regard as a subalgebra of its bidual (or enveloping W^*-algebra) B''. Let Q be the set of all $x \in B''$ such that $a \times b \in B$ for all $a, b \in B$ (that is, Q is the set of *quasi-*

multipliers of B). The *quasi-strict uniformity* is that induced by the family of all semi-norms $Q \ni x \to \|a^* xa\|$, where a runs through B. Let B' be the dual of B (so B'' is the dual of B'). Let Y be the unit sphere of B', and let $\mathscr{G}$ be the T-class induced on Y relative to the relativization of the weak-* topology $\sigma(B', B)$ to Y. Let Z be the field of complex numbers and $\mathscr{Z}$ the T-class induced by the usual uniformity on Z. Let $F \doteq Q$ and define $\phi | F \times Y \to Z$ be letting $f\phi y \equiv y(f)$ for all $(f, y) \in F \times Y$.

COROLLARY 2.5. *Let $A \subset F$ be norm-bounded. Then A is relatively compact relative to the quasi-strict topology if and only if A is equicontinuous on Y.*

PROOF. We first note that, for each $y \in Y$, $A\phi y$ is bounded, and thus precompact in Z. Furthermore, since the closed balls in F about 0 (relative to the norm $\| \quad \|$ on B'') are complete for the quasi-strict uniformity ([5] Theorem 3), it follows that A is relatively compact relative to the quasi-strict topology if and only if A is precompact for the quasi-strict uniformity. Consequently Corollary 1.2 will follow from Theorem 1 once it has been established that $\mathscr{G}_\phi(A)$ is the T-class $\mathscr{R}$ induced by the relativization of the quasi-strict uniformity to A.

That $\mathscr{R} \subset \mathscr{G}_\phi(A)$ follows from [5] Theorem 7. Assume that there exists $\alpha | D \to A$ belonging to $\mathscr{G}_\phi(A)$ but not to $\mathscr{R}$. By replacing α with $i\alpha$ if necessary, we may (and shall) assume that α is Hermitian. Since $\alpha \notin \mathscr{R}$, there exist $b \in B$ and $\varepsilon > 0$ such that to each $d \in D$ corresponds $m(d) \geqq d$ satisfying

(8) $$\|b^* \alpha_d b - b^* \alpha_{m(d)} b\| > \varepsilon.$$

Since each $b^*(\alpha_d - \alpha_{m(d)})b$ is Hermitian, there exists a state $f_d \in Y$ such that

(9) $$|f_d(b^*(\alpha_d - \alpha_{m(d)})b)| = \|b^*(\alpha_d - \alpha_{m(d)})b\| \quad ([7]\ 3.2.2^*)$$

From Alaoglu's Theorem follows that $f | D \to Y$ has a subnet h which converges to some h in the unit ball of B' relative to the weak-* topology $\sigma(B', B)$. Let $s \equiv \sup\{\|x\| : x \in A\}$ and let $\delta > 0$ be arbitrary. Choose a positive unit vector v in B such that $\|bv - b\| \leqq \frac{\delta}{2\|b\| \cdot s}$. Hence, for each $d \in D$, $\|b(\alpha_d - \alpha_{m(d)})(bv - b)\| \leqq \delta$ so that

(10) $$|f_d(b^*(\alpha_d - \alpha_{m(d)})bv)| \geqq |f_d(b^*(\alpha_d - \alpha_{m(d)})b)| - \delta.$$

From (9), (10), and the Cauchy-Schwarz inequality for positive functionals follows

$$\begin{aligned} \|b^*(\alpha_d - \alpha_{m(d)})b\| - \delta &\leqq |f_d(b^*(\alpha_d - \alpha_{m(d)})bv)| \\ &\leqq [f_d([b^*(\alpha_d - \alpha_{m(d)})b]^2)]^{1/2} [f_d(v^2)]^{1/2} \end{aligned}$$

$$\leq \|f_d\|^{1/2} \| [b^*(\alpha_d-\alpha_{m(d)})b]^2\|^{1/2} [f_d(v^2)]^{1/2}$$
$$= \|b^*(\alpha_d-\alpha_{m(d)})b\| \cdot [f_d(v^2)]^{1/2} ;$$

hence (8) yields

$$\left(1-\frac{\delta}{\varepsilon}\right)^{1/2} \leqq f_d(v^2).$$

Consequently we have

$$\left(1-\frac{\delta}{\varepsilon}\right)^{1/2} \leqq h(v^2)$$

Since δ was arbitrary, and $\|v^2\| \leqq 1$, it follows that $\|h\|=1$. Thus $h \in Y$ and so $h \xrightarrow{\mathscr{Y}} h$. Since $\alpha \in \mathscr{G}_\phi(A)$, we have $\alpha\phi h \in Z$. But this is inconsistent with (9) and (8)! It follows that $\mathscr{G}_\phi(A) \subset \mathscr{R}$. Hence $\mathscr{G}_\phi(A)=R$. Q. E. D.

COROLLARY 2.6. Let $A \subset F$ be norm-bounded. Then A is relatively compact relative to the quasi-strict topology if and only if A is equicontinuous on the family of states in B' (relative to $\sigma(B', B)$).

PROOF. Same as for Corollary 1.2.

3. Sharpening equicontinuity

It is sometimes useful to know that equicontinuity on a smaller set implies equicontinuity on a larger.

THEOREM *Let F and W be a dual pair of Hausdorff locally convex linear topological spaces, with scalar field Z and canonical bilinear form $\langle , \rangle$. Let Y be a subset of W such that*

(*i*) *the closed convex hull $[Y]$ of $Y \cap \{0\}$ equals $\bigcup_{r \in [0,1]} rY$;*

(*ii*) *the balanced hull $[[Y]]$ of $[Y]$ is radial;*

(*iii*) *$rY \cap [[Y]] = \phi$ for all $r>1$.*

Let $S \subset Y$ satisfy

(*iv*) *$[Y]=[S]$ is compact;*

and let $\mathscr{S}$ be the T-class for S induced by the topology on Y (relativized from W). Let ϕ be the restriction of $\langle , \rangle$ to $F \times Y$ and ψ be the restriction of $\langle , \rangle$ to $F \times S$. Let $A \subset F$ satisfy

(*v*) $s \equiv \sup \{|\langle x, t \rangle| : x \in A,\ t \in Y\} < \infty$;

(*vi*) *A is an E-set (relative to ψ).*

Then $A\phi$ is equicontinuous on Y.

PROOF. Assume $A\phi$ is not equicontinuous on Y. Then there exist a net $\kappa | D' \to Y$ convergent to some $k \in Y$, *a net* $\alpha | D' \to A$, and some $\varepsilon > 0$ such that

(11) $$|\langle\alpha_d,\ \kappa_d\rangle-\langle\alpha_d,\ k\rangle|>3\varepsilon \text{ for all } d\in D'.$$

From (v), (vi), and Theorem 1 follows that $\mathcal{G}_{\Psi}(A)$ is precompact; thus there exists a function $m|D\to D'$ such that $\alpha\circ m$ is a subnet of α belonging to $\mathcal{G}_{\Psi}(A)$.

We claim that, for each $b\in D$,

(12) $$|\langle\alpha_{m(d)}-\alpha_{m(c)},\ \kappa_{m(e)}\rangle|\geqq\varepsilon \text{ for some } c,d,e\geqq b.$$

Assume that the claim is false. Then there exists $b\in D$ such that

(13) $$|\langle\alpha_{m(d)}-\alpha_{m(c)},\ \kappa_{m(w)}\rangle|<\varepsilon \text{ for all } d,c,w\geqq b.$$

Since $\kappa\circ m$ converges to k, it follows from (13) that

(14) $$|\langle\alpha_{m(d)}-\alpha_{m(c)},\ k\rangle|<\varepsilon \text{ for all } d,c\geqq b.$$

Let $w\geqq b$ be such that $|\langle\alpha_{m(b)},\ \kappa_{m(w)}-k\rangle|<\varepsilon$; then (11) yields

$$3\varepsilon<|\langle\alpha_{m(w)},\ \kappa_{m(w)}-k\rangle|\leqq|\langle\alpha_{m(b)}-\alpha_{m(w)},\ k\rangle|$$
$$+|\langle\alpha_{m(w)}-\alpha_{m(b)},\ \kappa_{m(w)}\rangle|+|\langle\alpha_{m(b)},\ \kappa_{m(w)}-k\rangle|\leqq$$

(by (14) and (13)) $\varepsilon+\varepsilon+\varepsilon=3\varepsilon$.

This is absurd, which establishes (12).

From (12) follows that there exists a cofinal subset H of the product directed set $D\times D$ and a subnet $h|H\to Y$ of κ such that $|\langle\alpha_{m(d)}-\alpha_{m(c)},\ h_{(d,c)}\rangle|\geqq\varepsilon$ for all $(c,d)\in H$. Thus, defining the net $x|H\to F$ by letting $x_{(c,d)}\equiv\alpha_{m(d)}-\alpha_{m(c)}$ for all $(c,d)\in H$, we obtain

(15) $$|\langle x_e,\ h_e\rangle|>\varepsilon \text{ for all } e\in H.$$

From (ii), (iii), and the Hahn-Banach Theorem follows that there exists a linear functional $f|W\to Z$ such that

(16) $$f(k)=1=\sup\{|f(b)|:b\in[[Y]]\}.$$

Let Y° be the absolute polar of Y relative to the dual pair consisting of F and W. Then Y° is also the polar of $[[Y]]$ and there exists a sequence $\omega|N\to Y^\circ$ such that

(17) $$|\langle\omega_n,\ k\rangle-f(k)|<\frac{1}{n^2} \text{ for all } n\in N$$

([8] 1.5). From (16), (17), and the fact that $Y^\circ=[[Y]]^\circ$, (18) $\langle\omega_n,\ k\rangle>1-\dfrac{1}{n^2}$ and $|\langle\omega_n,\ c\rangle|\leqq 1$ for all $n\in N$, $c\in[[Y]]$.

Let $\mathcal{C}$ be the family of all functions $g|\Delta(g)\to[0,1]$ such that the domain $\Delta(g)$ is a finite subset of S and $\sum_{y\in\Delta(g)} g(y)=1$. For $g\in\mathcal{C}$ we shall write g' for the element $\sum_{y\in\Delta(g)} g(y)\cdot y$ of $[Y]$. From (iv) follows that for each $d\in H$, there exists a net $g^d|G(d)\to C$ such that $(g^d)'$ converges to h_d; hence by (15) there exists $\gamma(d)\in G(d)$ such that

(19) $$|\langle x_d,\ (g^d)_t'\rangle| > \frac{2\varepsilon}{3} \text{ for all } t \geqq \gamma(d).$$

Let G be the Cartesian product of H with the product $\underset{d\in H}{X}\, G(d)$, and define $g|G\to\mathscr{C}$ by letting $g_{(d,\gamma)} \equiv (g^d)_{\gamma(d)}$ for all $(d,\gamma)\in G$. By [4] 2.9, it follows that

(20) g' converges to k in W.

Let $n\in N$ satisfy

(21) $$n > 6s/\varepsilon.$$

From (18) and (20) follows that there exists $e(n)\in G$ such that

(22) $$|\langle \omega_n,\ g_t'\rangle| > 1-\frac{1}{n^2} \text{ for all } t\geqq e(n).$$

Let $c\in H$ be arbitrary and let $\gamma\in \underset{d\in D}{X}\, G(d)$ be as in (19). The set $M\equiv\{(n,\ v)\in N\times G: n>6s/\varepsilon,\ v\geqq(c,\gamma),$ and $v\geqq e(n)\}$ is cofinal in the product directed set $N\times G$ and so is a directed set. For $\nu=(n,v)\in M$, define $B(\nu)\equiv\{y\in\Delta(g_v): |\langle\omega_n,\ y\rangle| < 1-\frac{1}{n}\}$, $\Gamma(\nu)\equiv\Delta(g_\nu)/B(\nu)$, and $r(\nu)\equiv\sum_{y\in B(\nu)} g_\nu(y)$. We have, from (22) and (18),

$$1-\frac{1}{n^2} < \sum_{y\in B(\nu)} |\langle\omega_n,\ g_v(y)y\rangle| + \sum_{y\in\Gamma(\nu)} |\langle\omega_n,\ g_v(y)y\rangle| <$$
$$\left(1-\frac{1}{n}\right)\sum_{y\in B(\nu)} g_v(y) + \sum_{y\in\Gamma(\nu)} g_v(y)$$
$$=\left(1-\frac{1}{n}\right)\cdot r(\nu) + (1-r(\nu)) = 1-\frac{r(\nu)}{n};$$

consequently

(23) $$r(\nu) < \frac{1}{n} \text{ for all } \nu\in M,\ \nu=(n,v).$$

From (v) we see that, for all $(d,c)\in H$ and $y\in S$,

(24) $$|\langle x_{(d,c)},\ y\rangle| \leqq |\langle\alpha_{m(d)},\ y\rangle + |\langle\alpha_{m(c)},\ y\rangle \leqq 2s.$$

For $\nu=(n,v)\in M$ with $v=(w,\tau)\in G$, we have

$$\sum_{y\in\Gamma(\nu)} g_v(y)\cdot|\langle x_w,\ y\rangle| \geqq \left|\sum_{y\in\Gamma(\nu)} g_v(y)\langle x_w,\ y\rangle\right| \geqq$$
$$|\langle x_w,\ g_v'\rangle| - \sum_{y\in B(\nu)} g_v(y)|\langle x_w,\ y\rangle| \geqq$$

(by (19) and (24))

$$\frac{2\varepsilon}{3} - 2s\cdot r(\nu) \geqq$$

(by (23) and (21))

$$\frac{2\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3};$$

consequently, there exists $y_\nu\in\Gamma(\nu)$ such that

(25) $$|\langle x_w,\ y_\nu\rangle| > \varepsilon/3$$

By the definition of $\Gamma(\nu)$, we have

(26) $$|\langle\omega_n,\ y_\nu\rangle|\geqq 1-\frac{1}{n} \text{ for all } \nu=(n,v)\in M.$$

From (iv) follows that there exists a function $p|P\to M$ such that $y\circ p$ is a subnet of y with a limit $b\in[Y]$. From (26) follows that, for each $n>6s/\varepsilon$, $|\langle\omega_n,\ y\circ p\rangle|\geqq 1-\frac{1}{n}$ eventually; hence

(27) $$|\langle\omega_n,\ b\rangle|\geqq 1-\frac{1}{n}.$$

Assume $b\notin Y$. By (iii) there exists $t\in]0,1[$ and $c\in Y$ such that $b=tc$. Then, for $n>6s/\varepsilon+(1-t)^{-1}$, (27) yields

$$|\langle\omega_n,\ c\rangle|=\frac{1}{t}|\langle\omega_n,\ b\rangle|\geqq\frac{1}{t}\left(1-\frac{1}{n}\right)>\frac{1}{t}(1-(t-1))=1$$

which violates (18). It follows that b is in Y, and so $y\circ p$ belongs to the T-class $\mathcal{S}$.

Since $\alpha\circ m$ belongs to $\mathcal{G}_\phi(A)$, the net sending each $(d,p)\in D\times P$ to $\alpha_{m(d)}\phi y_{p(p)}=\langle\alpha_{m(d)},\ y_{p(p)}\rangle$ is Cauchy in Z. Consequently the net sending each $(e,p)\in H\times P$ to $\langle x_e,\ y_{p(p)}\rangle$ converges to 0 in Z. In particular, there exists $e_0\in H$ and $p_0\in P$ such that

(28) $$|\langle x_e,\ y_{p(p)}\rangle|<\varepsilon/3 \text{ for all } e\geqq e_0 \text{ and } p\geqq p_0.$$

Choosing $p\geqq p_0$ such that $w\geqq e_0$ where $p(p)=(n,v)$ and $v=(w,\tau)$, (28) becomes

(29) $$|\langle x_w,\ y_{p(p)}\rangle|<\varepsilon/3.$$

But (29) and (25) are inconsistent. Consequently $A\phi$ is equicontinuous on Y. Q.E.D.

EXAMPLE 3.2. Let B, B', B'', Q, Z, and $\mathcal{Z}$ be as in Example 2.4. Let P be the family of pure states on B, and K the family of states.

COROLLARY 3.3. *Let $A\subset Q$ be norm-bounded. The following are equivalent:*
(i) A is relatively compact in the quasi-strict topology;
(ii) for each net f in P which $\sigma(B',B)$-converges to some $f\in K$, $a(f)$ converges to $a(f)$ uniformly for $a\in A$.

PROOF. Let $W\equiv B'$, endowed with the weak topology $\sigma(B',Q)$. Let $S\equiv P$, $Y=K$, and let $F\equiv Q$. Note that $[K]$ is the set of all positive functionals in the unit ball of B' and that $[[K]]$ is a subset of the closed unit ball of B'. That conditions (i), (ii), (iii), and (v) of Theorem 3.1 hold is evident. It follows from Alaoglu's Theorem that $[K]$ is $\sigma(B',B)$-compact. Since $S\cup\{0\}$ is the set of extreme points for $[K]$, the Krein-Milman Theorem implies that the convex hull of $S\cup\{0\}$ is dense in $[K]$ relative to the topology $\sigma(B',B)$.

Thus, condition (iv) of Theorem 3.1 holds.

It follows from Theorem 3.1 that (ii) is equivalent to (30) A is equicontinuous on K relative to $\sigma(B',B)$. That (30) is equivalent to (i), follows from Corollary 2.6. Q.E.D.

EXAMPLE 3.4. Let B, B', B'', E, Z, and $\dot{K}$ be as in Example 2.4. In general Q may not be an algebra, but the set $M\equiv\{x\in B'' : xB\cup Bx\subset B\}$ of all *multipliers* of B constitutes a sub C^*-algebra of B''. For each $a\in B$, let n_a be the seminorm on M'.

$$n_a(x)\equiv\|ax\|+\|xa\| \text{ for all } x\in M.$$

The *strict uniformity* is that induced by the family $\{n_a : a\in B\}$ of seminorms—we write κ for the corresponding topology.

We note that, on the unit ball M_1 of M,

(31) κ is metrizible if B is separable;

(32) κ is the compact-open topology if B is the family $C_0(X)$ of all continuous, complex-valued functions vanishing at ∞ on a Hausdorff locally compact space X.

LEMMA 3.5. *Let $F\equiv K$ and let Y be a subset of M containing a multiple of an approximate identity x for B such that $\|x\|\to 1$. Let Y be the* T*-class on Y induced by κ. Let $\phi|F\times Y\to Z$ be defined by $\phi(f,x)\equiv x(f)$ for all $f\in F$ and $x\in Y$. Then $\mathcal{Q}_\phi(F)$ is the T-class R induced by the relativization of $\sigma(B', B)$ to F.*

PROOF. That $\mathcal{R}\subset\mathcal{Q}_\phi(F)$ follows from [6] Theorem 3.5. That $\mathcal{Q}_\phi(F)\subset\mathcal{R}$ follows from the proof of [5] Theorem 8. Q.E.D.

LEMMA 3.6. *The group U of unitary elements of M is closed relative to the topology κ.*

PROOF. Let μ be a net in U κ-convergent to some $u\in M$, and let $x\in B$ be arbitrary. Then

$$\|u^*ux-x\|\leqq\|(u^*-\mu^*)ux\|+\|\mu^*(u-\mu)x\|+\|\mu^*\mu x-x\|\leqq$$
$$\|x^*u^*(u-\mu)\|+\|(u-\mu)x\|+0\to 0$$

and, analogously $uu^*x-x=0$ as well. Thus $u^*u=uu^*$ is the identity and so $u\in U$ Q.E.D.

COROLLARY. 3.7. *Let B be a separable C^*-algebra. Let U be the group of unitary elements of M. The following are equivalent:*

(i) A is a relatively compact subset of K relative to the relativization of $\sigma(B'$,

B) to K;

(ii) for each net μ in U convergent to some $u\in U$ relative to the strict topology κ, $\mu(f)$ converges to $u(f)$ uniformly for $f\in A$.

PROOF. Since B is separable, there exists a commutative sub C^*-algebra C of B and a sequence x of positive unit vectors in C such that x is an approximate identity for B: that is, x converges to the identity d of B relative to κ. Let $N\equiv\{x\in M : xC\cup Cx\subset C\}$ and let $N_1=M_1\cap N$. Let X be the set of non-zero algebra homomorphisms of C into Z and $\wedge\ |N\to C(X)$ the extended Gelfand Transform: $\hat{x}(\phi)\equiv\phi(x)$ for all $x\in C$, $\phi\in X$.

Claim: the restriction of $\wedge$ to N_1 is a homomorphism when N_1 bears κ and the image bears the compact open topology. In fact, the continuity of $\wedge$ is a simple consequence of the Gelfond Theory. Suppose a is a net in N_1 such that $\hat{a}$ converges to $\hat{a}$ in the compact open topology (for $a\in N_1$). Let $b\in B$ and $\varepsilon>0$ be arbitrary. Choose $n\in N$ such that $\|b-bx_n\|+\|b-x_nb\|<\varepsilon/8$. Then

$$\begin{aligned}n_b(\alpha-a)=\|b(\alpha-a)\|(\alpha-a)b\|\leqq\|(b-bx_n)(\alpha-a)\|\\ +\|bx_n(\alpha-a)\|+\|(\alpha-a)(b-x_nb)\|+\|(\alpha-a)x_nb\|\leqq\\ \|b-bx_n\|\cdot\|\alpha-a\|+\|b\|\cdot\|x_n(\alpha-a)\|+\|(\alpha-a)\|\cdot\|b-x_nb\|\\ +\|(\alpha-a)x_n\|\cdot\|b\|\leqq\varepsilon/4+\|b\|\cdot\|\hat{x}_n(\hat{\alpha}-\hat{a})\|\\ +\varepsilon/4+\|(\hat{\alpha}-\hat{a})\hat{x}_n\|\cdot\|b\|\end{aligned}$$

which is less that ε eventually. Hence α κ-converges to a, so $\wedge|N_1$ is a homomorphism.

The set $L\equiv\{x_n : n\in N\}\cup\{d\}$ is the range of a convergent sequence, and so is κ-compact. Thus $\hat{L}$ is compact in $C(X)$ relative to the compact-open topology. By corollary 2.3, $\hat{L}$ is equicontinuous. Let Δ be the unit circle of the complex plane. Since the map $\Delta\times[0,1/2]\ni(z,\ t)\longrightarrow(z+t)(1+zt)^{-1}$ is continuous, it follows that the family $\{(z+f)(1+zf)^{-1} : f\in\frac{1}{2}\hat{L},\ z\in\Delta\}$ is equicontinuous, and hence relatively compact in the compact-open topology. Thus the set $S\equiv\left\{\left(z+\frac{1}{2}x_n\right)\left(1+\frac{1}{2}zx_n\right)^{-1} : n\in N,\ z\in\Delta\right\}$ is a κ-relatively compact subset of U.

It is known that the restriction of κ to M agrees the Mackey Topology $\tau(M, B')$ ([9] Theorem II and Corollary 2.3). It follows from Krein's Theorem ([8] 11.5) that the closed colnvex hull $[S]$ of $S\cup\{0\}$ is κ-compact. Let $F\equiv K$. Let W be the linear span in M of the closed balanced hull of $[S]$, endowed with the relativization of κ. Let Y be the set of all $x\in[S]$ such that $rx\notin[S]$ for all $r>0$. Then the closed balanced hull $[[Y]]$ of Y is a subset of M_1.

That conditions (i), (ii), (iii), (iv) and (v) of Theorem 3.1 hold is evident. It follows from Theorem 3.1 and Lemma 3.6 that (3.7, ii) implies that A is equicontinuous on Y.

For each $n\in N$, Let $r_n\in\left[\frac{1}{2}, 1\right]$ be maximal such that $r_n x_n\in[Y]$. Let r_m be a convergent subsequence of r with limit $r\in\left[\frac{1}{2}, 1\right]$. Then $r_m x_m$ κ-converges to rd and $r_m x_m$ is a sequence in Y.

Consequently

(33) Y contains a multiple of an approximate identity

$$\frac{r_m}{r}x_m \text{ for } B.$$

It follows from (33), Lemma 3.5, and Theorem 2.1 that (3.7, i) is equivalent to

(34) A is equicontinuous on Y.

It follows from Theorem 3.1 and Lemma 3.6 that (34) is equivalent to (3.7. ii). Q.E.D.

REFERENCES

[1] E. Binz and H. H. Keller, ***Funktionräme in der Kategorie der Limesräume***, Ann. Acad. Sci. Fenn. Ser. A I 383(1966) 1—21.

[2] H.R. Fisher, ***Limesräume***, Math. Ann. 137(1959) 269—303.

[3] R.J. Gazik and D.C. Kent, ***Regular completion of Cauchy spaces via function algebras***, Bull. Austral. Math. Soc 11(1974) 77—88.

[4] J.E. Kelley, ***General Topology***, Van Nostraud, Princeton, N.J., 1965.

[5] K. McKennon, ***Strict topology and Cauchy structure of the sprectrum of a C^*-algebra***, Gen. Top. and its Applications 5(1975) 249—262.

[6] K. McKennon, ***Multipliers, positive functionals, positive-definite functions, and Fourier-Stieltjes transforms***, Mem. Am. Math. Soc. 111(1971).

[7] G.K. Pedersen, ***C^*-algebras and their Automorphism Groups***, Academic Press, London, 1979.

[8] H.H. Schaefer, ***Topological Vector Spaces***, MacMillan Co., New York, 1966.

[9] D. Taylor, ***The strict topology for double centralizer algebras***, Trans. A.M.S. iso (1970) 633—643.