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# INDEFINITE KAEHLERIAN MANIFOLDS WITH VANISHING BOCHNER CURVATURE TENSOR*

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## Introduction

As a complex analogue to the Weyl conformal curvature tensor, Bochner and Yano [3], [13] (see also Tachibana [11]) introduced a Bochner curvature tensor in a Kaehlerian manifold. The study of vanishing Bochner curvature tensor with constant scalar curvature were made by Kubo [4], Matsumoto [5], Tachibana and Liu [12], Yano and Ishihara [14] and so on. One of which done by Matsumoto and Tanno [6] obtained the following interesting result:

THEOREM M.-T. *Let $M$ be a Kaehlerian manifold with constant scalar curvature whose Bochner curvature tensor vanishes. Then $M$ is a space of constant holomorphic sectional curvature or a locally product of two spaces of constant holomorphic sectional curvatures $c(\geqq 0)$ and $-c$. Thus, $M$ is locally symmetric.*

On the other hand, a systematic study of indefinite Kaehlerian manifolds has been made by Aiyama, Ikawa, Kwon and Nakagawa [1], Barros and Romero [2], Montiel and Romero [7] and Romero [9], [10].

The main purpose of the present paper is to study above theorem in an indefinite Kaehlerian manifold. Our main result is appeared in section 3.

## 1. The Bochner curvature tensor in an indefinite Kaehlerian manifold

We begin by recalling fundamental properties on indefinite Kaehlerian manifold. Let $M$ be a connected indefinite Kaehlerian manifold of complex dimension $n$. Then $M$ is equipped with a parallel almost complex structure $J$ and an indefinite Riemannian metric $g$ which is J-Hermitian. For the indefinite Kaehlerian structure $(g, J)$, it follows that $J$ is integrable and the index of $g$ is even, say $2s$ $(0\leqq s\leqq n)$. A local unitary frame field $\{E_i\}=\{E_a, E_{a^*}\}$ can be chosen, where

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$E_{a^*}=JE_a$. Here and in the sequel the indices $a, b, \cdots$run over the range $\{1, \cdots, n\}$, and $i, j, \cdots$the range $\{1, \cdots, n, 1^*, \cdots, n^*\}$. This is a complex linear frame which is orthogonal with respect to the Kaehlerian metric, namely, $g(E_i, E_j)=\varepsilon_i\delta_{ij}$, where $\varepsilon_i=\pm 1$. Its dual frame field $\{w_i\}=\{w_a, w_{a^*}\}$ on $M$ such that $w_i(E_j)=\varepsilon_i\delta_{ij}$ are linearly independent. Thus, the Kaehlerian metric $g$ of $M$ can be expressed locally by $g=2\Sigma\varepsilon_i w_i\otimes w_i$. Associated with the frame field $\{E_i\}$, there exist linear 1-forms $w_{ij}$, which are usually *connection forms* on $M$, such that they satisfy the sturcture equation of $M$:

$$(1.1)\qquad dw_i+\Sigma\varepsilon_j w_{ij}\Lambda w_j=0,\ w_{ij}+w_{ji}=0,$$

$$(1.2)\qquad dw_{ij}+\Sigma\varepsilon_k w_{ik}\Lambda w_{kj}=\Omega_{ij},\ \Omega_{ij}=-\frac{1}{2}\Sigma\varepsilon_k\varepsilon_m R_{ijkm}w_k\Lambda w_m,$$

where $\Omega_{ij}$ (resp. $R_{ijkm}$) denotes the curvature forms (resp. components of the Riemannian curvature tensor $R$) on $M$.

For the almost complex sturcture $J$, we let $J=\Sigma\varepsilon_i\varepsilon_j J_{ij}E_i\otimes w_j$. Then the equation $J^2=-I$, $I$ being the identity tensor, is equivalent to

$$(1.3)\qquad \Sigma\varepsilon_k J_{ik}J_{kj}=-\varepsilon_i\delta_{ij},\ J_{ij}+J_{ji}=0.$$

Let $S$ be the Ricci tensor of $M$ and let $S_{ij}$ be components of $S$ with respect to the frame $\{E_i\}$. It is given by $S_{ij}=\Sigma\varepsilon_k R_{kijk}$. The scalar curvature $r$ on $M$ is given by $r=\Sigma\varepsilon_i S_{ii}$.

In an indefinite Kaehlerian manifold $M$, we put

$$(1.4)\qquad L_{ji}=-\frac{1}{2(n+2)}S_{ji}+\frac{r}{8(n+1)(n+2)}\varepsilon_j\delta_{ji},$$

$$(1.5)\qquad M_{ji}=-\Sigma\varepsilon_r L_{jr}J_{ir}=-\frac{1}{2(n+2)}H_{ji}+\frac{r}{8(n+1)(n+2)}J_{ji},$$

$$(1.6)\qquad H_{ji}=-\Sigma\varepsilon_r S_{jr}J_{ir}.$$

The Bochner curvature tensor $B$ with components $B_{kjih}$ relative to a real coordinate system in an indefinite Kaehlerian manifold has been given by (Tachibana [13])

$$(1.7)\qquad \begin{aligned} B_{kjih}=&R_{kjih}+\varepsilon_k(\delta_{kh}L_{ji}-\delta_{ki}L_{jh})+\varepsilon_j(\delta_{ji}L_{kh}-\delta_{jh}L_{ki})\\ &+J_{kh}M_{ji}-J_{ki}M_{jh}+M_{kh}J_{ji}-M_{ki}J_{jh}-2(M_{kj}J_{ih}+J_{kj}M_{ih}). \end{aligned}$$

When the Bochner curvature tensor vanishes identically, by means of (1.4) and (1.5) the following equation is dervied from (1.7):

$$(1.8)\qquad \begin{aligned} R_{kjih}=&\frac{1}{2(n+2)}\{\varepsilon_k(\delta_{kh}S_{ji}-\delta_{ki}S_{jh})+\varepsilon_j(\delta_{ji}S_{kh}-\delta_{jh}S_{ki})\\ &+J_{kh}H_{ji}-J_{jh}H_{ki}+H_{kh}J_{ji}-H_{jh}J_{ki}-2(H_{kj}J_{ih}+J_{kj}H_{ih})\}\\ &-\frac{r}{4(n+1)(n+2)}\{\varepsilon_k\varepsilon_j(\delta_{kh}\delta_{ji}-\delta_{jh}\delta_{ki})+J_{kh}J_{ji}-J_{jh}J_{ki}-2J_{kj}J_{ih}\}. \end{aligned}$$

A holomorphic plane spanned by $u$ and $Ju$ which is the tangent vector at any point in an indefinite Kaehlerian manifold $M$ is non-degenerate if and only if it contains some $v$ such that $g(v,v)=0$. The manifold $M$ is said to be *of constant holomorphic sectional curvature* $c$, if all nondegenerate holomorphic planes have the same constant sectional curvature $c$, A. complete, simply connected and connected indefinite Kaehlerian manifold $M$ is called an *indefinite complex space form*, which is denoted by $M^n_s(c)$, provided that it is of complex dimension $n$, of index $2s$ and of constant holomorphic sectional curvature $c$. There are three kinds of types about indefinite complex space forms [2], [7]: an indefinite complex projective space, an indefinite complex Euclidean space or an indefinite hyperbolic space, according as $c$ is positive, zero or negative. The indefinite Riemannian manifold is said to be ***Einstein***, if the Ricci tensor $S$ is expressed by $S_{ij}=\lambda\varepsilon_i\delta_{ij}$. In this case, (1.6) is reduced to $H_{ji}=\lambda J_{ji}$. Substituting these into the right hand side of (1.8), the Riemannian curvature tensor of $M$ is given by

$$R_{kjih}=\frac{c}{4}\{\varepsilon_k\varepsilon_j(\delta_{kh}\delta_{ji}-\delta_{ih}\delta_{kj})+J_{kh}J_{ji}-J_{jh}J_{ki}-2J_{kj}J_{ih}\}$$

for some constant $c$. Therefore $M$ is of constant holomorphic sectional curvature $c$(cf. [2]). This fact is well known in the definite case (see [11]).

## 2. Vanishing Bochner curvature tensor

Let $M$ be an indefinite Kaehlerian manifold of complex dimension $n$ and of index $2s$, and $(g, J)$ its Kaehlerian structure. For the almost complex sturcture $J$, components $J_{ijk}$ and $J_{ijkm}$ of the covariant differentials $\nabla J$ of $J$ and $\nabla^2 J$ of $\nabla J$are respectively defined by

$$(2.1)\qquad \begin{aligned}&\Sigma\varepsilon_k J_{ijk}w_k=dJ_{ij}-\Sigma\varepsilon_k(J_{kj}w_{ki}+J_{ik}w_{kj}),\\ &\Sigma\varepsilon_m J_{ijkm}w_m=dJ_{ijk}-\Sigma\varepsilon_m(J_{mjk}w_{mi}+J_{imk}w_{mj}+J_{ijm}w_{mk}).\end{aligned}$$

Since $J$ is parallel, $J_{ijk}$ and $J_{ijkm}$ vanish identically. Thus, by making use of the last two equations the Ricci formula for $J_{ij}$ gives rise to

$$\Sigma\varepsilon_r R_{mkjr}J_{ri}=\Sigma\varepsilon_r R_{mkir}J_{rj},$$

which imples

$$(2.2)\qquad \Sigma\varepsilon_r\varepsilon_s R_{rjis}J_{rs}=\Sigma\varepsilon_r S_{jr}J_{ir}.$$

Thus, the tensor $H_{ji}$ defined by (1.6) is skew-symmetric and hence

$$(2.3)\qquad S_{ji}=\Sigma\varepsilon_r\varepsilon_s S_{rs}J_{jr}J_{is}$$

because of (1.3). By taking account of (2.2) and the first Bianchi formula it is seen that

(2.4) $$S_{ji}=-\frac{1}{2}\Sigma\varepsilon_s\varepsilon_r\varepsilon_t R_{jtsr}J_{sr}J_{it}.$$

For the Ricci tensor $S$, the Riemannian curvature tensor $R$, component $S_{ijk}$ (resp. $R_{kjihm}$) of the covariant differential $\nabla S$(resp. $\nabla R$) of $S$(resp. R) are defined by

(2.5) $$\Sigma\varepsilon_k S_{ijk}w_k=dS_{ij}-\Sigma\varepsilon_k(S_{kj}w_{ki}+S_{ik}w_{kj}),$$

(2.6) $$\Sigma\varepsilon_m R_{kihmj}w_m=dR_{kjih}-\Sigma\varepsilon_m(R_{mjih}w_{mk}+R_{kmih}w_{mj}+R_{kjmh}w_{mi}+R_{kjim}w_{mh}).$$

Substituting (2.4) for the Ricci tensor in the right hand side of (2.5) and making use of (2.1), we find

$$S_{ijk}=-\frac{1}{2}\Sigma\varepsilon_r\varepsilon_s\varepsilon_t R_{jtsrk}J_{sr}J_{it},$$

which together with the second Bianchi formula yield

(2.7) $$S_{ikj}-S_{ikj}=\Sigma\varepsilon_r\varepsilon_s S_{krs}J_{jr}J_{is}.$$

Components of the covariant differential $\nabla B$ of the Bochner curvature tensor $B$ can be defined by a similar to (2.6). Then by a straight forward computation, we can prove (cf. [4], [11])

(2.8) $$\Sigma\varepsilon_r B_{kjirr}=\frac{n+2}{n}A_{kji},$$

where we have put

(2.9) $$A_{kji}=S_{ijk}-S_{ikj}+\frac{1}{4(n+1)}\{\varepsilon_i(\delta_{ki}r_j-\delta_{ji}r_k)+\Sigma\varepsilon_t r_t(J_{ki}J_{jt}-J_{ji}J_{kt}+2J_{kj}J_{it})\},$$

$r_k$ being components of the covariant differential $\nabla r$ of the scalar curvauture $r$.

In the sequel we suppose that the indefinite Kaehlerian manifold $M$ has vanishing Bochner curvature tensor. Then, by means of (2,7) and (2.8), the equation (2,9) turns out to be

$$\Sigma\varepsilon_r\varepsilon_s S_{krs}J_{jr}J_{is}=\frac{1}{4(n+1)}\{\varepsilon_i(\delta_{ij}r_k-\delta_{ki}r_j)+\Sigma\varepsilon_t r_t(J_{ji}J_{kt}-J_{ki}J_{jt}-2J_{kj}J_{it})\},$$

Accordingly, by properties of the Kaehlerian structure, it follows that

(2,10) $$S_{ijk}=\frac{1}{4(n+1)}\{\varepsilon_j\delta_{jk}r_k+\varepsilon_i\delta_{ik}r_j+2\varepsilon_i\delta_{ij}r_k-\Sigma\varepsilon_t(J_{kj}J_{it}r_t+J_{ki}J_{jt}r_t)\},$$

For the sake of brevity, a tensor $S^2_{ij}$ and a function $S_2$ on $M$ are introduced as follows:

(2.11) $$S^2_{ij}=\Sigma\varepsilon_r S_{ir}S_{rj},\quad S_2=\Sigma\varepsilon_i S^2_{ii}.$$

Components $S_{ijkm}$ of the covariant differetial $\nabla^2 S$ of $\nabla S$ are defined by

(2.12) $$\Sigma\varepsilon_m S_{ijkm} w_m = dS_{ijk} - \Sigma\varepsilon_m (S_{mjk} w_{mi} + S_{imk} w_{mj} + S_{ijm} w_{mk}).$$

Differentiating (2.5) exteriorly and ulitirizing (2,12), we get

$$2\Sigma\varepsilon_k\varepsilon_m S_{ijkm} w_m / w_k = \Sigma\varepsilon_r\varepsilon_k\varepsilon_m (S_{rj} R_{rikm} + S_{ir} R_{rjkm} + S_{ir} R_{rjkm}) w_k / w_m,$$

which is equivalent to the formula of Ricci,

$$S_{ijkm} - S_{ijmk} = -\Sigma\varepsilon_r (R_{mkir} \dot{S}_{rj} + R_{mkjr} S_{ir}).$$

From this and (2.10), it is easy to see that

(2.13) $$-\Sigma\varepsilon_r (R_{mkir} S_{rj} + R_{mkjr} S_{ir})$$
$$= \frac{1}{4(n+1)} \{\varepsilon_i (\delta_{ki} r_{jm} - \delta_{mi} r_{jk}) + \varepsilon_j (\delta_{jk} r_{im} - \delta_{jm} r_{ik})$$
$$- \Sigma\varepsilon_t (J_{kj} J_{it} r_{mt} - J_{mj} J_{it} r_{tk} + J_{ki} J_{jt} r_{tm} - J_{mi} J_{it} r_{tk})\},$$

where $r_{ji}$ denotes components of the covariant differential of $\nabla^2 r$,

Multiplying $\varepsilon_k\varepsilon_k J_{jj}$ to this and summing for $k$ and $j$ and making use of (2,2), (2.3) and (2.5), we can easily verify that

$$\Sigma\varepsilon_t H_{mt} S_{it} + \Sigma\varepsilon_t\varepsilon_s R_{mtsi} H_{ts} = \frac{1}{4(n+1)} \Sigma\varepsilon_t \{J_{mt} r_{ti} - (2m-1) J_{it} r_{mt} + r_{tt} J_{im}\},$$

where $H_{ji} = -\Sigma\varepsilon_t S_{jt} J_{it}$. Since the tensor $H_{ji}$ is skew-symmetric, it follows that $\Sigma\varepsilon_t (J_{it} r_{mt} + J_{mt} r_{it}) = 0$. By summing up (2.13) with respect to indices $m$ and $j$ and taking account of the last equations, we see that

$$S^2_{ij} - \Sigma\varepsilon_k\varepsilon_h R_{kjih} S_{kh} = \frac{1}{4(n+1)} (\varepsilon_i\delta_{ij}\Delta r - 2nr_{ij}),$$

where $\Delta r = \Sigma\varepsilon_t r_{tt}$. Because of (1.3), (1,8) and (2,3), the last relationship reduces to

(2.14) $$(n+2)(\Delta r\varepsilon_i\delta_{ij} - 2nr_{ij}) = 4n(n+1) S^2_{ij} - 2nrS_{ij} + \{r^2 - 2(n+1)S_2\}\varepsilon_i\delta_{ij}.$$

## 3. Results

First of all we prove

LEMMA 1. *Let $M$ be an indefinite Kaehlerian manifold with vanishing Bohner curvature tensor. Then $S_2$ is constant if and only if $r$ is.*

PROOF. "If" part is evident because of (2.10). Thus, we are going to prove only if part. Transvecting (2.10) with $\varepsilon_i\varepsilon_j S_{ij}$ and using (1.3), we find

(3.1) $$2\Sigma\varepsilon_s S_{ks} r_s + rr_k = 0.$$

By differentiating this exteriorly and making use of (2,5) and (2.10), we have

$$(3.2) \qquad (2n+5)r_j r_k + \varepsilon_j \delta_{jk} (\Sigma \varepsilon_i r_i)^2 - \varepsilon_s \varepsilon_t (J_{js} r_s)(J_{kt} r_t) + 2(n+1)(\{2\Sigma \varepsilon_i S_{jt} r_{tk} + r_{jk}\} = 0.$$

On the other hand, applying $\varepsilon_i r_i$ to (2.14) and taking account of (3.1), we obtain

$$(n+2)(r_j \Delta r - 2n\Sigma \varepsilon_t r_t r_{tj}) = (n+1)\{(n+1)r^2 - 2S_2\} r_j,$$

which together with (3.1) gives

$$4(n+2)\left\{-\frac{1}{2} r r_j \Delta r - 2n\Sigma \varepsilon_t \varepsilon_s S_{jt} r_s r_{st}\right\} = -r\{(n+1)r^2 - 2S_2\} r_j.$$

Therefore, the last two equations yield

$$\Sigma \varepsilon_t \varepsilon_s S_{jt} r_s r_{ts} + \frac{1}{2} r \Sigma \varepsilon_t r_t r_{jt} = 0.$$

Combining this with (3.2), we can easily verify that $r$ is constant everywhere. This completes the proof of the lemma.

Under the same assumptions as those stated in Lemma 1, the relationship (2.10) means that $M$ has parallel Ricci tensor and hence so is for $H_{ij}$ because $J$ is parallel. Thus, the equation (1.8) tells us that $M$ is locally symmetric. Thus we have

THEOREM 2. *Let $M$ be an indefinite Kaehlerian manifold with vanishing Bochner curvature tensor. The following statements are equivalent*:

(1) *$M$ has the constant scalar curvature*, (2) *$S_2$ is constant, where $S_2 = \Sigma \varepsilon_i S_{ii}^2$*, (3) *$M$ has the parallel Ricci tensor*, (4) *$M$ is locally symmetric*,

REMARK. It is easily checked that a deinite Kaehlerian manifold with vanishing Bochner curvature tensor and with constant semi-definite Ricci curvatures is Einstein.

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