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COMPACT REGULAR FRAMES AND THE SIKORSKI THEOREM

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Recall that, in Zemelo-Fraenkel Set Theory, the Sikorski Theorem (ST) by which every complete Boolean algebra is injective, follows from the Axiom of Choice (AC) and, in turn, implies the Boolean Ultrafilter Theorem (BUT), where the latter implication is strict (Bell [3]) while it is still unknown whether this also holds for the former. In fact, conflicting opinions have been expressed in the literature on this point. Thus, Luxemburg [10] conjectured that both implications are strict whereas Bell [3], after proving this for the second one, leans to the view that ST may actually be equivalent to AC.

This paper presents an attempt to find, within the context of frames and of topological spaces, an approach that might be helpful to resolve this question. Specifically, we establish the equivalence between ST, the condition that every deMorgan compact regular frames is injective in the category of all such frames, and the assertion that, for any extension $M \supseteq L$ of compact regular frames, there exist $s \in M$ maximal such that $x \vee s = e$(=top element) implies $x = e$, for all $x \in L$. Analogously, in terms of topological spaces, we obtain that ST is equivalent to the conjunction of BUT and the condition that, for any continuous onto map $f: X \rightarrow Y$ between compact Hausdorff spaces, there exists a closed subspace $Z \subseteq X$ minimal such that $f[Z] = X$, and the conjunction of BUT with the condition that the extremally disconnected compact Hausdorff spaces are the projectives in the category of all such spaces.

In preparation for these results, we need to establish a number of facts concerning compact regular frames, most notably the existence of maximal essential extensions, which we call Gleason envelopes. It should be emphasized that, if BUT is assumed, all facts required here are immediate consequences of familiar results, especially those of Gleason [5] on projective compact Hausdorff spaces, in view of the duality between compact regular frames and compact Hausdorff spaces which then follows because BUT implies the spatiality of such frames (Banaschewski [1]). The point here is to establish these facts without the assumption of any choice principle.

Concerning Gleason envelopes, it should be added that a good part of what is presented here is contained in the much more general results of Johnstone [7, 8] on the Gleason cover of a topos. However, it may still be of some merit to have a direct lattice theoretic treatment of this important special case which avoids the extensive topos theoretic machinery required in [7, 8].

Even though, in the end, this paper fails to establish the exact relation between ST and AC it is hoped that the various equivalents of ST offered here may yet prove to be of some use in determining the precise strength of ST.

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1. Some basic facts

First we recall some general terminology. A *frame* is a complete lattice satisfying the distribution law $x\wedge\bigvee x_i=\bigvee x\wedge x_i$ for binary meet $\wedge$ and arbitrary join $\bigvee$. A map $h: L\to M$ between frames is called a *homomorphism* whenever it preserves finite meets and arbitrary joins, including the empty cases which means preservation of the unit e (=top) and the zero 0 (=bottom). In any frame, an element c is called *compact* if $c\leq\bigvee x_i$ $(i\in I)$, for any family of elements $(x_i)_{i\in I}$, implies $c\leq\bigvee x_k$ $(k\in E)$ for some finite subset $E\subseteq I$. A frame itself is called compact whenever its unit is compact. For any elements x and a in a frame, $x\prec a$ (x is rather below a) means that $x\wedge t=0$ and $a\vee t=e$ for some t. If $a=\bigvee x(x\prec a)$ for all its elements, a frame is called *regular*. **KRegFrm** will be the category of all compact regular frames and their homorphisms. For general background on frames and compact regular frames see Johnstone [9].

The following is a familiar characterization of embeddings, that is, one-one homomorphisms, in **KRegFrm**.

LEMMA 1. *For any $h: L\to M$ in KRegFrm, the following are equivalent:*
(1) *h is an embedding.*
(2) *For all $x\in L$, $h(x)=0$ implies $x=0$.*
(3) *For all $x\in L$, $h(x)=e$ implies $x=e$.*

PROOF. (1)$\Rightarrow$(2). Obvious.

(2)$\Rightarrow$(3). If $h(x)=e$ then also $h(z)=e$ for some $z \dashv x$, by regularity and compactness. Then, for t such that $z \wedge t=0$ and $x \vee t=e$, one has $h(t)=h(z) \wedge h(t) =h(z \wedge t)=0$, hence $t=0$ by hypothesis and therefore $x=e$.

(3)$\Rightarrow$(1). If $h(x)=h(y)$, take any $z \dashv x$ and t such that $z \wedge t=0$ and $x \vee t=e$. Then $e=h(x \vee t)=h(y \vee t)$, hence $y \vee t=e$ by hypothesis, and therefore $z \leq y$. Be regularity, this shows $x \leq y$, and then $x=y$ by symmetry.

Below, we let $\uparrow s=\{x \in L \mid x \geq s\}$ for any element s of any frame L. Note this is again a compact regular frame if L is, and the map $(\cdot) \vee s: L \to \uparrow s$ taking x to $x \vee s$ is a frame homomorphism.

COROLLARY. *Any $h: L \to M$ in **KRegFrm** has a decomposition*

$$L \xrightarrow[(\cdot)\vee s]{} \uparrow s \xrightarrow[\bar{h}]{} M$$

where $s=\bigvee h^{-1}\{0\}$ and the homomorphism $\bar{h}$ induced by h is an embedding.

PROOF. Since $h(x)=h(x \vee s)$ for any $x \in L$, by the definition of s, we only have to show that $\bar{h}$ is one-one. If $x \geq s$ in L then $\bar{h}(x)=0$ implies $x \leq s$ and hence $x=s$, the zero of $\uparrow s$. By Lemma 1 this shows $\bar{h}$ is one-one.

REMARK. Given any map $h: L \to M$ in **KRegFrm** and $s=\bigvee h^{-1}\{0\}$, consider the $(x, y) \in L$ such that

$$(x, y) \leq (s, s) \text{ or } (x, y) \in h^{-1}\{e\} \times h^{-1}\{e\}.$$

These clearly constitute a subframe K of $L \times L$, and a more detailed analysis shows that K is regular. It follows from this that *h is one-one whenever it is monic*: if $p, q: K \to L$ are the homomorphisms induced by the two projections $L \times L \to L$ then $hp=hq$, hence $p=q$ and therefore $s=0$.

For later purposes we have to know that **KRegFrm** has pushouts. This is a special case of the general fact that **KRegFrm** *is closed under all colimits in the (cocomplete!) category* **Frm** *of all frames*. To see this we only have to recall (Banaschewski-Mulvey [2]) that **KRegFrm** is coreflective in **Frm**; indeed, the coreflection of any frame L is the largest regular subframe **K**L of the frame **J**L of ideals of L, with coreflection map **K**$L \to L$ given by taking joins.

Of particular interest here will be the *essential embeddings* in **KRegFrm**, that is, the embeddings $h: L \to M$ such that, for any map $f: M \to K$ in **KRegFrm**, f is an embedding whenever fh is. Of course, this is a notion which plays a role in a large number of other categories. The following provides a useful characterization.

LEMMA 2. *An embedding $h: L \to M$ in **KRegFrm** is essential iff, for each $a \in M$, if $h(x) \leq a$ implies $x=0$, for all $x \in L$, then $a=0$.*

PROOF. ($\Rightarrow$) Consider any $a \in M$ such that $h(x) \leq a$ implies $x=0$, for all $x \in L$. Then, for the composite homomorphism

$$g: L \xrightarrow[h]{} M \xrightarrow[(\cdot)\vee a]{} \uparrow a,$$

$g(x)=a$ implies $h(x) \leq a$, hence $x=0$, and therefore g is one-one by Lemma 1. By essentialness, it follows that $(\cdot) \vee a$ is also one-one, showing that $a=0$.

($\Leftarrow$) Given $f: M \to N$ in **KRegFrm** such that fh is an embedding, consider any $a \in M$ for which $f(a)=0$. Then, for any $x \in L$, $h(x) \leq a$ implies $fh(x) \leq f(a)=0$, hence $fh(x)=0$ and therefore $x=0$. It follows then by hypothesis that $a=0$, and by Lemma 1 this shows f is an embedding. Hence, h is an essential embedding.

COROLLARY. *If a composite $fg: L \to K \to M$ of embedding in **KRegFrm** is essential then g is essential.*

PROOF. Consider any $a \in K$ such that $g(x) \leq a$ implies $x=0$ for all $x \in L$. Then, $fg(x) \leq f(a)$ implies $g(x) \leq a$ and therefore $x=0$ for any $x \in L$. Since fg is essential, Lemma 2 implies that $f(a)=0$ and hence $a=0$. This shows g is essential, by Lemma 2.

REMARK. It should be noted that the above arguments are constructive in that they do not involve the Law of the Excluded Middle nor any choice principle. If one is not concerned about that point then the essential embeddings are alternatively characterized as those embeddings $h: L \to M$ such that, for each $a > 0$ in M, there exist $x \in L$ for which $0 < h(x) \leq a$.

2. The gleason envelope

Recall the result of Gleason [5] that, for any compact Hausdorff space X, there exists an extremally disconnected such space Y with an irreducible onto map $Y \to X$, where *extremally disconnected* means that open sets have open closures and *irreducible* that no closed proper subspace of Y is mapped onto X. This fact about the category **KHaus** of compact Hausdorff spaces only requires BUT so that, if this is assumed, one obtains the corresponding dual result in

KRegFrm by means of the duality between **KHaus** and **KRegFrm** which also follows from BUT.

It is a consequence of Johnstone [7] that this dual result can actually be obtained directly and, moreover, constructively and hence in particular without the use of any choice principle. Here, a purely lattice theoretic proof of this fact is presented. Note there is one difference between the approach in [7] and ours: we stress the role of essential extensions which has no counterpart in [7].

A $Y \to X$ of the type described above is called the Gleason cover of X; hence the dual construct in **KRegFrm** will be called the *Gleason envelope*.

A frame L is called *deMorgan*(also: extremely disconnected, or: Stone algebra) whenever the identity $x^{*} \vee x^{**} = e$ holds in L, where $(\ )^{*}$ stands for pseudo-complementation, that is, x^{*} is the largest element in L disjoint from x. For verious details concerning deMorgan frames see Johnstone [9]. Note in particular that the identity $x^{*} \vee x^{**} = e$ is equivalent to the deMorgan law $(x \wedge y)^{*} = x^{*} \vee y^{*}$. Also, the frame of open sets of a topologicalspace X is deMorgn iff X is extremely disconnected.

Here we are interested in the deMorgan $L \in$ **KRegFrm**. Such L are generated, as frames, by the Boolean algebra CL of complemented elements of L because $x \dashv a$ implies $x^{**} \dashv a$ for any x, a in L and therefore

$$a = \bigvee_{x \dashv a} x = \bigvee_{x \dashv a} x^{**}$$

for each $a \in L$. This makes L isomorphic to the $\mathbf{J}(CL)$ of ideals of CL. Furthermore, CL is complete, the join of any $S \subseteq CL$ being $(\bigvee s)^{**}$. In particular, this means, for any Boolean algebra B, that B is complete whenever $\mathbf{J}B$ is deMorgan since $B \cong C(\mathbf{J}B)$. Conversely, for any complete Boolean algebra B, $\mathbf{J}B$ is deMorgan since completeness implies, for any ideal J of B, that $J^{*} = \downarrow(\sim \bigvee J)$ ($\sim$the complementation in B), and $(\downarrow a^{*} = \downarrow(\sim a)$ holds anyway for each $a \in B$. Finally, one easily sees that the correspondences $L \rightsquigarrow CL$ and $B \rightsquigarrow \mathbf{J}B$ between compact regular deMorgan frames and complete Boolean algebras are functorial, providing a category equivalence between the full subcategory of **KRegFrm** given by the deMorgan frames and the category of complete Boolean algebras and all Boolean homomorphisms between them.

Actually, the link between compact regular frames and complete Boolean algebras is even closer than that: we shall now show that each $L \in$ **KRegFrm** has an essential embedding into a compact deMorgan frame.

For any frame L, it is well-known that $L_{**} = \{x \in L \mid x = x^{**}\}$ is a complete

Boolean algebra (Glivenko [6]). If L is compact regular, put $GL=\mathbf{J}(L_{**})$ and define

$$\gamma : L \to GL \text{ by } \gamma(a)=J_a=\{x\in L_{**} \mid x \dashv a\}.$$

J_a is indeed an ideal in L_{**}: clearly $y\leq x\in J_a$ implies $y\in J_a$ and x, $y\in J_a$ implies $x\vee z\dashv a$ and hence also $(x\vee z)^{**}\dashv a$ where $(x\vee z)^{**}$ is the join of x and z in L_{**}.

Since $a=\bigvee x(x\dashv a)=\bigvee x^{**}(x\in a)$ for each $a\in L$, the map $GL\to L$ by taking joins is left inverse to γ and hence γ is one-one. Moreover, it is a frame homomorphism: J_0 and J_e are clearly the bottom and top, respectively, of GL. Also, $J_a\cap J_b=J_{a\wedge b}$ since $x\dashv a$, b implies $x\dashv a\wedge b$. Further, for updirected $S\subseteq L$, $J_{\bigvee s}=\bigsqcup J_a$ $(a\in S)$ since $x\dashv \bigcup s$ means that $x\wedge t=0$ and $t\vee\bigvee s=e$ for some t, and by compactness one then also has $t\vee a=e$ for some $a\in S$ so that $x\in a$. Finally, if $x\dashv a\vee b$ for $x\in L_{**}$, and hence $x\wedge t=0$ and $a\vee b\vee t=e$ with suitable t, then also (compactness again) $u\vee v\vee t=e$ for some $u\dashv a$ and $v\dashv b$ which may be taken such that $u=u^{**}$ and $v=v^{**}$, hence $x=(x\wedge u)\vee(x\wedge v)$ where $x\wedge u\dashv a$, $x\wedge v\dashv b$ and $x\wedge u$, $x\wedge v\in L_{**}$. This shows $J_{a\vee b}\subseteq J_a\vee J_b$ and hence equality.

Thus γ is a frame embedding. consider any $J\in GL$ such that $J_a\subseteq J$ implies $a=0$ for all $a\in L$. Then, for any $x\in J$, $J_x\subseteq J$, hence $x=0$, and therefore J is the zero ideal. In all, this proves

PROPOSITION 1. *For each $L\in$ **KRegFrm**, $\gamma : L\to GL$ is an essential embedding into a deMorgan frame in **KRegFrm**.*

An obvious consequence of the rusult is that any $L\in$ **KRegFrm** which has no proper essential extension is deMorgan since $\gamma : L\to GL$ must be an isomorphism for such L. This proves the easy part of the following

PROPOSITION 2. *A compact regular frame is deMorgan iff it has no proper essential extension.*

PROOF. We have to show $(\Rightarrow)$. Thus, let $L\in$ **KRegFrm** be deMorgan and $M\supseteq L$ any essential extension, that is, the identical embedding $L\to M$ is essential. We define $\lambda : M\to L$ by

$$\lambda(a)=\bigvee\{x\in L \mid x\leq a\},$$

so that $\lambda(a)$ is the largest element in L below $a\in M$. We establish a number of fact concerning λ in order to prove $a\in L$ for each $a\in M$.

In the following, $(\ )^{*}$ will be pseudocomplementation in M and $(\ )^{\#}$ that in L.

First, $a\wedge\lambda(a)^{*}=0$ for any $a\in M$. If $x\leq a\wedge\lambda(a)^{*}$ for any $x\in L$ then also $x\leq\lambda(a)$ by the defintion of λ and hence $x=0$; thus $a\wedge\lambda(a)^{*}=0$ by Lemma 2.

Next, $a^{*}=\lambda(a)^{*}$ for each $a\in M$ since $\lambda(a)^{*}\leq a^{*}$ because $a\wedge\lambda(a)^{*}=0$ whereas $a^{*}\leq\lambda(a)^{*}$ because $\lambda(a)\leq a$.

Further, $x^{\#}=x^{*}$ for each $x\in L$. Obviously, $x^{\#}\leq x^{\#}$ since $L\subseteq M$. On the other hand, if $a\wedge x=0$ for any $a\in M$ then trivially $\lambda(a)\wedge x=0$, hence $\lambda(a)^{\#}\wedge x^{\#}=e$ because L is deMorgan, and therefore $a\leq x^{\#}$ since $a\wedge\lambda(a)^{\#}\leq a\wedge\lambda(a)^{*}=0$, as previously proved. Now, for $a=x^{*}$ this yields $x^{*}\leq x^{\#}$, and hence equality.

Finally, if $c\dashv a$ for any $a, c\in M$ then also $c^{**}\leq a$ where $c^{**}=\lambda(c)^{**}=\lambda(c)^{\#\#}$ by the two preceding steps. Since

$$a=\bigvee_{c\dashv a}c=\bigvee_{c\dashv a}c^{**}$$

this means

$$a=\bigvee_{c\dashv a}\lambda(c)^{\#\#}$$

which says that $a\in L$ as desired.

We conclude this section with the following uniqueness result:

PROPOSITION 3. *As an essential embedding in* **KRegFrm** *into a deMorgan frame,* $\gamma: L\to GL$ *is unique up to a unique isomorphism.*

PROOF. For any $L\in$ **KRegFrm**, consider any essential deMorgan extension $M\supseteq L$ in **KRegFrm**. As before, we define $\lambda: M\to L$ by $\lambda(a)=\bigvee x(a\geq x\in L)$. Since $M\supseteq L$ is essential, we again have that $\lambda(a)$ is dense in a, for each $a\in M$. Also, as prviously, $(\ )^{\#}$ will be pseudocomentation in L, as opposed to $(\ )^{*}$ in M. Now put

$$I_a=\{x\in L_{**}\,|\,x\dashv a\}\quad(\mathrm{a}\in M).$$

This is an ideal in L_{**}: First note that, for any $z\in L$ and $c\in M$, if $z\wedge c=0$ then $z\wedge\lambda(c)=0$ trivially, hence $z^{**}\wedge\lambda(c)=0$ by the properties of pseudocomplementation, and finally $z^{\#\#}\wedge c=0$ by density. This shows, for any $z\in L$ and $a\in M$, that $z\dashv a$ implies $z^{\#\#}\dashv a$. Hence, for any $x, y\in I_a$, $(x\vee y)^{\#\#}\dashv a$, since $x\vee y\dashv a$, showing that the join of x and y in L_{**} again belongs to I_a.

We claim that the map $a\rightsquigarrow I_a$ is a frame homomorphism $M\to GL$, which then obviously extends the embedding $\gamma: L\to GL$. That this map preserves zero and unit, binary meet and updirected join is seen exactly the same way as the corresponding properties of γ; concerning binary join, the argument here has to be more subtle. If $x \dashv a \vee b$ for any $x \in L_{**}$ and $a, b \in M$ then, as previously noted, $x=(u \wedge x) \vee (v \wedge x)$ for $u \dashv a$ and $v \dashv b$ in M, which may be taken suh that $u=u^{**}$ and $v=v^{**}$. To get inside L, consider $\lambda(u \wedge x) \leq u \wedge x$ and $\lambda(v \wedge x) \leq v \wedge x$. Since these are dense in M it follows that $\lambda(u \wedge x) \vee \lambda(u \wedge x) \leq x$ is dense in L and hence

$$(\lambda(u \wedge x) \vee (v \wedge x))^{\#\#} = x^{\#\#} = x.$$

This shows, for $y=\lambda(u \wedge x)^{\#\#}$ and $z=\lambda(u \wedge x)^{\#\#}$, that x is the join of y and z in L_{**}. Moreover, $y \leq u$ and $z \leq v$: consider any $t \leq y \wedge u^{*}$ in L. Then $t \wedge \lambda(u \wedge x) = 0$ since $t \leq u^{*}$, hence also $t \wedge y = 0$ since $\lambda(u \wedge x)$ is dense in y, and therefore $t=0$ since $t \leq y$. M being an essential extension of L, this implies $y \wedge u^{*}=0$, and hence $y \leq u^{**}=u$. The argument equally applying to z, this now shows we have y and z in L_{**} such that $y \dashv a$, $z \dashv b$ and x is their join in L_{**}. Hence $I_{a \vee b} = I_a \vee I_b$ as desired.

It now follows further that the frame homomorphism $M \to GL$ by $a \rightsquigarrow I_a$ is an embedding since it extends γ, and γ is an essential embedding. Finally, the image of M in GL has GL as essential extension and hence must be all of GL by Proposition 2. Thus, our map $M \to GL$ is an isomorphism.

For the additional uniqueness property, let $M \supseteq L$ as before and $h: M \to M$ any automorphism leaving each $x \in L$ fixed. Consider, first, any $a=a^{**}$ in M. Then, for $\lambda(a)$ defined as before, $a^{**}=\lambda(a)^{**}$ by density and hence $a=\lambda(a)^{**}$. Therefore

$$h(a)=h(\lambda(a)^{**})=h(\lambda(a))^{**}=\lambda(a)^{**}=a,$$

showing that h also leaves each complemented element of M fixed. Finally, since

$$a=\bigvee_{x \dashv a} x = \bigvee_{x \dashv a} x^{**}$$

it follows that $h(a)=a$ for all $a \in M$, as claimed.

REMARK. Since any endomorphism $h: GL \to GL$ such that $h\gamma=\gamma$ must be an automorphism, by the arguments in the above proof, it further follows that the only such h is the identity map.

As a consequence of Proposition 3 we note the following.

COROLLARY. *For any $L \in$ **KRegFrm**, $\gamma: L \to GL$ is the largest essential extension of L.*

PROOF. Given any essential extension $M \supseteq L$, $L \subseteq M \to GM$ is an essential embedding of L into a deMorgan frame in **KRegFrm**, and the unique isomorphism from GM to GL extending γ then provides an embedding of M into GL extending γ.

In many situations, most typically for modules over a ring but also for a variety of other notions, there is an intimate relationship between essential extensions and injectivity. In fact, the result in those cases is that an object is injective iff it has no proper essential extensions, and every object has an injective essential extension, called its injective hull. The arguments required to establish this usually involve AC; moreover, in certain cases, such as that of abelian groups, the use of AC is known to be essential (Blass [4]).

We want to determine what is required to make the same results hold in **KRegFrm**. For this, consider the following condition which expresses the reduction of arbitrary extensions to essential ones:

REE. For any extension $M \supseteq L$ in **KRegFrm**, there exist $s \in M$ maximal such that $x \vee s = e$ implies $x = e$, for all $x \in L$.

Also, recall that, by general terminology, $L \in$ **KRegFrm** is *injective* iff, for any embedding $h : M \to N$ and any homomorphism $f ; M \to L$ there exists a homomorphism $g : N \to L$ such that $gh = f$.

Now we have

PROPOSITION 4. *The compact regular deMorgan frames are exactly the injectives in KRegFrm iff REE.*

PROOF. ($\Leftarrow$) Without any assumption, the essential embedding $\gamma : L \to GL$ has a left inverse whenever L is injective, which makes γ an isomorphism and therefore L deMorgan. Thus, we have to derive from REE that, conversely, $L \in$ **KRegFrm** is injective if it is deMorgan. For this, consider first any extension $M \supseteq L$ and let $s \in M$ be as in REE. Then the map $L \to \uparrow s$ taking x to $x \vee s$ is an essential embedding: if $s < a$ there exists $x < e$ in L such that $x \vee a = e$ and hence also $z \prec\!\!\prec x$ in L for which $z \vee a = e$, and if $t \in L$ is such that $z \wedge t = 0$ and $x \vee t = e$ then $t \leq a$ so that $t \vee s \leq a$ whereas $s < t \vee s$ since $t \leq s$ implies $x \vee s = e$ and therefore $x = e$, a contradiction. It now follows from Proposition 2 that $L \to \uparrow s$ is an isomorphism, showing there exist $h : M \to L$ such that $h|L = id_L$. We express this result by saying L is an *absolute retract* in **KRegFrm**.

In particular, this applies to the two-element frame **2** which is evidently deMorgan, but that in turn implies **2** is injective. To see this, consider the diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\xi"'] & M \arrow[d, "(\cdot)\vee s"] \\ \mathbf{2} \arrow[r] & \{z\in M \mid z\geq s\} \end{tikzcd}$$

where h is any embedding, ξ any homomorphism as indicated, $s=\bigvee h(x)$ $(\xi(x)=0)$, and the bottom map the unique homomorphism. Then, $s<e$ since $\bigvee\xi^{-1}\{0\}<e$ and h is an embedding. Hence the bottom map is an embedding, and since **2** is an absolute retract there exist $g:\{z\in M \mid z\geq s\}\to\mathbf{2}$ left inverse to it. As a result, $f:M\to\mathbf{2}$ such that $f(x)=g(x\vee s)$ is the desired map such that $fh=\xi$: if $\xi(a)=0$ then $h(a)\leq s$, hence $h(a)\vee s=s$, and therefore

$$fh(a)=g(h(a)\vee s)=g(s)=0,$$

whereas $\xi(a)=1$ implies $h(a)\vee s=e$ and hence $fh(a)=1$.

Now consider any diagram

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "f"'] & N \arrow[d, "v"] \\ L \arrow[r, "u"'] & K \end{tikzcd}$$

where L is deMorgan, f a given arbitrary homomorphism, h a given embedding, and K obtained by pushout. Then, take any $a<e$ in L. Since the unique homomorphism $\mathbf{2}\to\uparrow a$ is an embedding, it follows there is a homomorphism $\xi: L\to\mathbf{2}$ such that $\xi(a)=0$, and since **2** is injective there further exist $\zeta:N\to\mathbf{2}$ such that $\zeta h=\xi f$. Then, by the property of pushouts, there exist $\sigma:K\to\mathbf{2}$ for which $\sigma u=\xi$ and $\sigma v=\zeta$. In particular, $\sigma u(a)=0$ which implies $u(a)<e$. By Lemma 1, it follows u is an embedding, and since we already know L is an absolute retract u has a left inverse $w:K\to L$. Then, $g=wv:N\to L$ is the desired map such that $f=hg$.

($\Rightarrow$) Given any extension $M\supseteq L$ in **KRegFrm**, let $h:M\to GL$ be an extension of $\gamma: L\to GL$ by injectivity and

$$M\xrightarrow[(\cdot)\vee s]{}\uparrow s\xrightarrow[\bar{h}]{}\mathrm{GL}$$

the decomposition of h as in the Corollary of Lemma 1. Then, for any $x\in L$, $x\vee s=e$ implies $\gamma(x)=h(x)=h(x\vee s)=e$ and hence $x=e$. Thus, it remains to show s is maximal among the elements with this properry. Now, the homomorphism $L \to \uparrow s$ induced by $(\cdot)\vee s$ is an essential embedding by the Corollary of Lemma 2, and hence, for any $a \in \uparrow s$, if $x \vee s \leq a$ implies $x=0$, for all $x \in L$, then $a=s$. Now, consider any $a \in \uparrow s$ for which it is still the case that $x \vee a = e$ implies $x=e$, for all $x \in L$. Then, if $x \vee s \leq a$ for any $x \in L$, take any $y \prec x$ in L and, correspondingly, $t \in L$ such that $y \wedge t = e$ and therefore $t=e$ by the hypothesis on a. This implies $y=0$, and since $y \prec x$ was arbitrary we have $x=0$ by regularity. Thus, $x \vee s \leq a$ implies $x=0$, for all $x \in L$, and therefore $a=s$. This establishes the maximality of s.

REMARK. AC clearly implies REE: for any given extension $M \supseteq L$ in **KRegFrm**, the set $F \subseteq M$ of all elements t such that $x \vee t = e$ implies $x=e$, for each $x \in L$, is closed under joins of chains, in fact even under joins of arbitrary updirected sets, by compactness, and hence Zorn's Lemma ensures the existence of maximal elements in F.

3. The Sikorski Theorem

We are now ready to relate the Sikorski Theorem to conditions in **KRegFrm** and in **KHaus.** The first result of this type is

PROPOSITION 5. *ST iff the compact regular deMorgan frames are the injectives in* ***KRegFrm.***

PROOF. ($\Rightarrow$) For any embedding $h : L \to M$ in **KRegFrm** with L deMorgan, the composite map $\gamma h : L \to M \to GM$ induces a homomorphism $f : CL \to C(GM)$ between the Boolean algebras of complemented elements of L and GM, respectively, which has a left inverse $g : C(GM) \to CL$ by the injectivity of the complete Boolean algebra CL. Hence, one has the commuting diagram

$$
\begin{tikzcd}
L \arrow[r, "h"] & M \arrow[r, "\gamma"] & GM \\
JCL \arrow[u, "\sigma"] \arrow[rr, shift left, "\bar{f}"] & & JC(GM) \arrow[u, "\tau"'] \arrow[ll, shift left, "\bar{g}"]
\end{tikzcd}
$$

where $\bar{f}$ and $\bar{g}$ are induced by f and g, respectively, and the vertical maps are the isomorphisms between the compact regular deMorgan frames and the ideal lattices of their lattices of complemented elements. Here, $\sigma\bar{g}\tau^{-1}\gamma$ is a left inverse of h since $\bar{g}$ is a left inverse of $\bar{f}$. This proves that every deMorgan $L\in$**KRegFrm** is an absolute retract, and by the first part of the proof of Proposition 4 it then follows that L is injective.

($\Leftarrow$) the category equivalence between complete Boolean algebras and compact regular de Morgan frames is actually a part of the more comprehensive equivalence between the category of all Boolean algebras and that of all zero-dimensional compact regular frames, that is, those $L\in$**KRegFrm** which are generated by their complemented elements. This larger equiqalence is again given by the ideal lattice functor in one direction and by the functor $L\rightsquigarrow CL$, taking the Boolean algebra of complemented elements, in the other. Hence, if A is a complete Boolean algebra then **J**$A\in$**KRegFrm** is deMorgan, hence injective in **KRegFrm** by the present hypothesis and thus, a fortiori, injective in the category of 0-dimensional compact regular frames which makes A itself an injective Boolean algebra.

Directly from the last proposition and Proposition 4 we obtain further:

PROPOSITION 6. *ST iff REE.*

Passing from frames to topological spaces, consider the following condition which expresses the reduction to irreducible maps in **KHaus**:

RIM. For any continuous onto map $f: X\to Y$ between compact Hausdorff spaces, there exists a closed subspace $Z\subseteq X$ minimal such that $f[Z]=Y$.

The relation between this condition and its frame version REE is given in

LEMMA 3. *REE iff BUT and RIM.*

PROOF. ($\Rightarrow$) By Proposition 6, REE implies ST and hence the injectivity of the two-element Boolean algebra, which is equivalent to BUT. Further, given any onto map $f: X\to Y$ in **KHaus** and the associated embedding $\mathbf{O}Y: \mathbf{O}Y\to\mathbf{O}X$ where $\mathbf{O}$ is the functor assigning the frame of open sets to a space, then the $S\in\mathbf{O}X$ provided by REE has as its complement a closed subspace $Z\subseteq X$ minimal such that $f[Z]=Y$. This follows from the observation that, for any closed subspace $T\subseteq X$ with complement $W\in\mathbf{O}X$, $f[T]=Y$ iff $f^{-1}(U)\cup W=X$ implies $U=Y$, for all $U\in\mathbf{O}Y$.

($\Leftarrow$) BUT implies the duality between **KRegFrm** and **KHaus.** Hence for any extension $M\supseteq L$ in **KRegFrm**, we can consider the dual $f: X\to Y$ where $M\cong\mathbf{O}Y$,

$L \cong \mathbf{O}X$ and f represents the inclusion map $L \to M$. Then, the complement of the closed subspace $Z \subseteq X$ provided by RIM determines an element $s \in M$ of the desired kind, again by the observation at the end of the previous paragraph.

As an immediate consequence of this lemma and the previous proposition we now have

PROPOSITION 7. *ST if BUT and RIM.*

For our final result, recall that Gleason [5] proves the following, using AC:

GT. The extremally disconnected compact Hausdorff spaces are exactly the projectives in **KHaus.**

The duality between **KHaus** and **KRegFrm** resulting from BUT then shows that Proposition 4 also implies the following

PROPOSITION 8. *ST iff BUT and GT.*

REMARK. There is the following analogue of Proposition 6 dealing directly with Boolean algebras: ST iff, for and extension $B \supseteq A$ of Boolean algebras, there exists an ideal $J \subseteq B$ maximal such that $A \cap J = 0$. I am indebted to W. A. J. Luxemburg for this observation who based it on the results of [11]. The present context indicates a direct proof in which the completion of a Boolean algebra parallels the role of the Gleason hull of a compact regular frame.

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