HONAM MATHEMATICAL JOURNAL
Volume 10. Number 1. August 1988

Computations of Some $\widetilde{K}_F(S^n)$ and $\widetilde{K}_F(P_3(\boldsymbol{R}))^*$

by

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In spite of that computing $\hat{K}$-groups of topological spaces is one of important works in K-theory, to do this is generally very hard. But, the followings have been computed in [2] and [3]:

$$\left.\begin{array}{lll} \hat{K}_{\boldsymbol{R}}(S^0)\cong\boldsymbol{Z}, & \hat{K}_{\boldsymbol{C}}(S^0)\cong\boldsymbol{Z} \text{ and } & \hat{K}_{\boldsymbol{R}}(P_2(\boldsymbol{R}))\cong\boldsymbol{Z}/4 \\ \hat{K}_{\boldsymbol{R}}(S^1)\cong\boldsymbol{Z}/2, & \hat{K}_{\boldsymbol{C}}(S^1)=0 & \hat{K}_{\boldsymbol{C}}(P_2(\boldsymbol{R}))\cong\boldsymbol{Z}/2 \\ \hat{K}_{\boldsymbol{R}}(S^2)\cong\boldsymbol{Z}/2, & \hat{K}_{\boldsymbol{C}}(S^2)\cong\boldsymbol{Z} & \\ \hat{K}_{\boldsymbol{R}}(S^3)=0, & \hat{K}_{\boldsymbol{C}}(S^3)=0 & \\ \hat{K}_{\boldsymbol{R}}(S^4)\cong\boldsymbol{Z}, & \hat{K}_{\boldsymbol{C}}(S^4)\cong\boldsymbol{Z} & \end{array}\right\} \qquad \text{(A)}$$

where $\boldsymbol{Z}$=the set of integers, $\boldsymbol{R}$=the set of reals, C=the set of complexes and $P_n(\boldsymbol{R})$ is the n-dimensional real projective space. Since $P_1(\boldsymbol{R})\approx S^1$, $\hat{K}_{\boldsymbol{R}}(P_1(\boldsymbol{R}))\cong\boldsymbol{Z}/2$ and $\hat{K}_{\boldsymbol{C}}(P_1(\boldsymbol{R})=0$.

The purpose of this note is to compute $\hat{K}$-groups $\hat{K}_F(S^6)$, $\hat{K}_F(S^6)$ and $\hat{K}_F(P_3(\boldsymbol{R}))$, where $F=\boldsymbol{R}$ or $\boldsymbol{C}$ (Theorem 4 and Theorem 6).

Throughout this note by a topological space we mean a connected and compact topological space without any statements. For a topological space X we shall put

$$\mathrm{Vec}_F(X)=\text{the class of all } F\text{-vector bundles over } X.$$

Then,

$$\Phi_F(X)=\mathrm{Vec}_F(X)/\cong$$

is an abelian monoid with the Whitney sum of vector bundles as the acditive operator

* The Present Studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1987.
Received April 30, 1988.

in $\Phi_F(X)$, where "$\cong$" means to be isomorphic between vector bundles over X. Let $K_F(X)$ be the ***symmetrization*** ([3]) of the abelian monoid $\Phi_F(X)$. Then $K_F(X)$ is abelian and it is called the *K-group* of X.

Let $\{p\}$ be a topological space consisting of only one point p, and let $X \longrightarrow \{p\}$ be the projection. This projection induces a group homomorphism

$$\mathbf{Z} \cong K_F(\{p\}) \longrightarrow K_F(X),$$

whose cokernel is denoted by $\hat{K}_F(X)$ which is called the *reduced* K-group of X. It follows that

$$K_F(X) \cong \mathbf{Z} \oplus \hat{K}_F(X)$$

([2], [3], [4]).

Let $\Phi_F^n(X)$ be the set of isomorphism classes of F-vector bundles of rank n over a topological space X. Taking the Whitney sum by trivial bundles enables us to define an inductive system of sets such that

$$\Phi_F^0(X) \longrightarrow \Phi_F^1(X) \longrightarrow \dots \longrightarrow \Phi_F^n(X) \longrightarrow \dots,$$

where $\Phi_F^r(X) \longrightarrow \Phi_F^{n+1}(X)$ is defined by $[E] \longmapsto [E] \oplus [\theta_n] \times [E \oplus \theta_n]$ and θ_n a trivial bundle of rank n over X. If we put

$$\varinjlim_n \Phi_F^n(X) = \Phi_F'(X)$$

then we have

$$\hat{K}_F(X) \cong \Phi_F'(X) \quad \cdots\cdots (B)$$

([3], [4]).

On the other hand there is the formula

$$\pi_{n-1}(GL_p(F))/\pi_0(GL_p(F)) \cong \Phi_F^p(S^n),$$

where $GL_p(F)$ is the general linear group of degree p over F([2], [3], [4]). Since

$$GL_p(\mathbf{R}) \approx O(p) \times \mathbf{R}^q \qquad (q = p(p+1)/2)$$

$$GL_p(C) \approx U(p) \times R^q \qquad (q=p^2),$$

where $O(p)$ = the orthogonal group of degree p over **R** and $U(p)$ = the unitary group of degree p over C, we have the following:

$$\begin{aligned}\Phi_R^p(S^n) &\cong \pi_{n-1}(GL_p(R))/\pi_0(GL_p(R)) \\ &\cong \pi_{n-1}(O(p)\times R^q)/\pi_0(O(p)\times R^q) \\ &\cong \pi_{n-1}(O(p))/\pi_0(O(p))\end{aligned}$$

and

$$\Phi_C^p(S^n) \cong \pi_{n-1}(U(p))/\pi_0(U(p)).$$

Moreover, since $\pi_0(O(p)) \cong Z/2$ and $\pi_0(U(p))=0$ it follows that

$$\Phi_R^p(S^n) \cong \pi_{n-1}(O(p))/Z/2,\ \Phi_C^p(S^n) \cong \pi_{n-1}(U(p)) \quad \cdots\cdots\cdots \text{(C)}.$$

Proposition 1. (i) If $p>i+1$ then $\pi_i(O(p)) \cong \pi_i(O(p+1))$, and hence $\varinjlim_m \pi_i(O(m)) \cong \pi_i(O(p))$.

(ii) If $p>\frac{i}{2}$ then $\pi_i(U(p)) \cong \pi_i(U(p+1))$, and hence $\varinjlim_m \pi_i(U(m)) \cong \pi_i(U(p))$.

Proof. Consider the locally trivial fibration

$$O(p) \longrightarrow O(p+1) \longrightarrow S^p \approx O(p+1)/O(p)$$

where $O(p) \longrightarrow O(p+1)$ is defined by $A(\in O(p)) \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix}$ ([2]). Thus we have the long homotopy exact sequence

$$\cdots \longrightarrow \pi_{i+1}(S^p) \longrightarrow \pi_i(O(p)) \longrightarrow \pi_i(O(p+1)) \longrightarrow \pi_i(S^p) \longrightarrow \cdots$$

([1]. Since if $i+1<p$ then $\pi_{i+1}(S^p)=0$ it follows that

$$\pi_i(O(p)) \cong \pi_i(O(p+1))$$

for all $p>i+1$. Suppose the inductive system

$$\begin{array}{ccccccc} O(p) & \longrightarrow & O(p+1) & \longrightarrow & O(p+2) & \longrightarrow \cdots\cdots \\ \Cup & & \Cup & & \Cup & \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} & \longmapsto & \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \longmapsto \end{array}$$

Then, this induces the inductive system

$$\pi_i(O(p)) \longrightarrow \pi_i(O(p+1)) \longrightarrow \pi_i(O(p+2)) \longrightarrow \cdots$$

for all $i \geq 0$. If $i+1 < p$ then we have

$$\pi_i(O(p)) \cong \pi_i(O(p+1)) \cong \pi_i(O(p+2)) \cong \cdots$$

and $$\varinjlim_m \pi_i(O(m)) \cong \pi_i(O(p)).$$

(ii) Similary, consider the locally trivial fibration

$$U(p) \longrightarrow U(p+1) \longrightarrow S^{2p+1} \approx U(p+1)/U(p)$$

([2]). From the exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{i+1}(S^{2p+1}) \longrightarrow \pi_i(U(p)) \longrightarrow \pi_i(U(p+1)) \longrightarrow \pi_i(S^{2p+1}) \longrightarrow \cdots$$

if $\frac{i}{2} < p$ then we have

$$\pi_i(U(p)) \cong \pi_i(U(p+1)).$$

From the inductive system

$$\pi_i(U(p)) \longrightarrow \pi_i(U(p+1) \longrightarrow \cdots$$

if $p > \frac{i}{2}$ then it follows that

$$\varinjlim_m \pi_i(U(m)) \cong \pi_i(U(p)). \quad ///$$

For $\varinjlim_m \pi_i(O(m))$ and $\varinjlim_m \pi_i(U(m))$ we have the following results ([3]) and Proposition 1):

Property 2. (i) If $p>i+1$ the

$$\pi_i(O(p)=\varinjlim_{m}\pi_i(O(m))\cong\begin{cases}\boldsymbol{Z}/2, & i\equiv 0 \bmod 8\\ \boldsymbol{Z}/2, & i\equiv 1 \bmod 8\\ 0, & i\equiv 2 \bmod 8\\ \boldsymbol{Z}, & i\equiv 3 \bmod 8\\ 0, & i\equiv 4,5,7 \bmod 8.\end{cases}$$

(ii) If $p>\frac{i}{2}$ then

$$\pi_i(U(p))=\varinjlim_{m}\pi_i(U(m))\cong\begin{cases}0, & i=\text{even}\\ Z, & i=\text{odd}\end{cases}$$

Proposition 3. The action of $\pi_0(GL_p(\boldsymbol{R}))\cong\pi_0(O(p))\cong\boldsymbol{Z}/2$ on $\pi_i(GL_p(\boldsymbol{R}))\cong\pi_i(O)p))$ is trivial if $p>i+1$.

Proof. (i) In case p is an odd. We can put

$$\boldsymbol{Z}/2\cong\pi_0(O(p))=\left\{\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix},\begin{pmatrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{pmatrix}\right\}$$

Recall that for each $\alpha\in\pi_0(O(p))$ and $[f]\in\pi_i(O(p))$

$$\alpha[f]=[\alpha\circ f(x)\circ\alpha^{-1}]\qquad {}^{\forall}x\in S^i$$

But it is clear that

$$\begin{pmatrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{pmatrix}\circ f(x)\circ\begin{pmatrix}-1 & & 0\\ & \ddots & \\ 0 & & -1\end{pmatrix}^{-1}=f(x)\qquad(x\in S^i)$$

Therefore we have

$$\pi_i(O(p))/\pi_0(O(p))\cong\pi_i(O(p))\qquad(p>i+1).$$

(ii) In case p is an even. At first we note that

$$\begin{array}{ccc}\pi_0(O(p)) & \longrightarrow & \pi_0(O(p+1))\\ \Cup & & \Cup\\ A & \longmapsto & \begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}\end{array}$$

is an isomorphism. It follows that for $p>i+1$

$$\begin{aligned}\pi_i(O(p))/\pi_0(O(p)) &\cong \pi_i(O(p+1))/\pi_0(O(p)) \\ &\cong \pi_i(O(p+1))/\pi_0(O(p+1)) \\ &\cong \pi_i(O(p+1)) \qquad \text{(by (i))} \\ &\cong \pi_i(O(p)) \qquad \text{(by Proposition 1).}\end{aligned}$$

Hence $\pi_0(O(p))$ $(p>i+1)$ acts on $\pi_i(O(p))$ to be trivial. ///

Theorem 4. $\hat{K}_{\boldsymbol{R}}(S^5) \cong \hat{K}_{\boldsymbol{R}}(S^6) \cong \hat{K}_{\boldsymbol{C}}(S^5) = 0$ and $\hat{K}_{\boldsymbol{C}}(S^6) \cong \boldsymbol{Z}$.

Proof. By (B), (C) and Proposition 3

$$\hat{K}_{\boldsymbol{R}}(S^n) \cong \varinjlim_m \pi_{n-1}(O(m))/\boldsymbol{Z}/2 \cong \varinjlim_m \pi_{n+1}(O(m))$$

and

$$\hat{K}_{\boldsymbol{C}}(S^n) \cong \varinjlim_m \pi_{n-1}(U(m)).$$

On the other hand, by Property 2,

$$\varinjlim_m \pi_4(O(m)) = \varinjlim_m \pi_5(O(m)) = \varinjlim_m \pi_4(U(m)) = 0$$

and

$$\varinjlim_m \pi_5(U(m)) \cong \boldsymbol{Z}.$$

Therefore, we get the following:

$$\hat{K}_{\boldsymbol{R}}(S^5) = \hat{K}_{\boldsymbol{R}}(S^6) = \hat{K}_{\boldsymbol{C}}(S^5) = 0, \text{ and } \hat{K}_{\boldsymbol{C}}(S^6) \cong \boldsymbol{Z}. \quad ///$$

For arbitrary topological spaces X, Y and a continuous function $f: X \longrightarrow Y$ there exists the Barratt-Puppe sequence

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow SX \xrightarrow{Sf} SY$$

([1]), where C_f is the mapping cone of f and SX is the reduced suspension of X.

In this case we have the exact sequence of $\hat{K}$-groups:

$$\hat{K}_F(SY)\longrightarrow\hat{K}_F(SX)\longrightarrow\hat{K}_F(C_f)\longrightarrow\hat{K}_F(Y)\longrightarrow\hat{K}_F(X)\cdots(\mathrm{D})$$

([2], [3], [4]). For $f\colon X\longrightarrow Y$ let M_f be the mapping cylinder of f, and let $i\colon Y\longrightarrow M_f$ be the inclusion. Put

$$C_f'=M_f/i(Y),$$

then there is an isomorphism

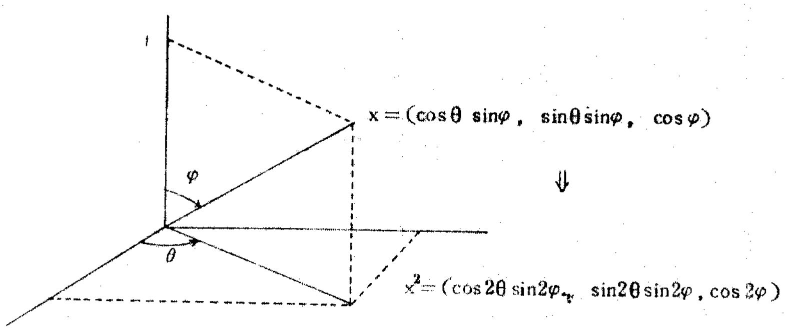
$$\hat{K}_F(C_f)\cong\hat{K}_F(C_f')\cdots\cdots(\mathrm{E})$$

Recall that $S^n/\sim\approx P_n(\boldsymbol{R})$ where for all $x\in S^n$ $x\sim\pm x$ and $n\geq 1$. Hence there is the projection $p\colon S^n\longrightarrow P_n(\boldsymbol{R})$.

Proposition 5. Let $f\colon S^2\longrightarrow P_2(\boldsymbol{R})$ be defined by the composition

$$\begin{array}{ccccc} S^2 & \longrightarrow & S^2 & \longrightarrow & P_2(\boldsymbol{R}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ x & \longmapsto & x^2 & \longmapsto & p(x^2) \end{array}$$

which is continuous, where



Then $C_f'\approx P_3(\boldsymbol{R})$.

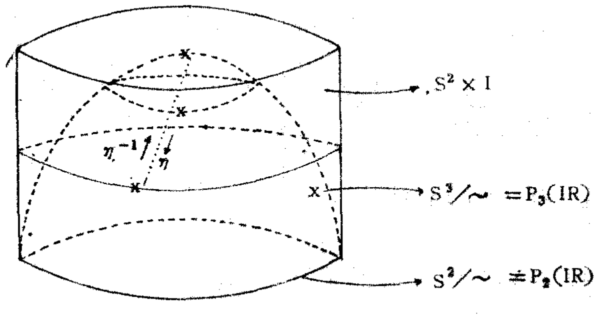
Proof. Note that

$$(S^2\times I\cup P_2(\boldsymbol{R}))/\sim/S^2\times 1=C_f',$$

where $(x,O)(\in S^2\times O)\sim f(x)(\in P_2(\boldsymbol{R}))$. Therefore

$$\eta:\ S^3/\sim = P_3(\boldsymbol{R})\approx C_f',$$

where η is the map denoting as in the following figure:



Theorem 6. $\hat{K}_{\boldsymbol{R}}(p_3(\boldsymbol{R}))\cong \boldsymbol{Z}/2\cong \hat{K}_{\boldsymbol{C}}(P_3(\boldsymbol{R}))$.

Proof. Using the continuous map f in Proposition 5 we have the Barratt-Puppe sequence

$$S^2\xrightarrow{f}P_2(\boldsymbol{R})\longrightarrow C_f\longrightarrow S^3\xrightarrow{Sf}S(P_2(\boldsymbol{R})).$$

From (D)

$$\hat{K}_F(S(P_2(\boldsymbol{R}))\longrightarrow \hat{K}_F(S^3)\longrightarrow \hat{K}_F(C_f)\longrightarrow \hat{K}_F(P_2(\boldsymbol{R})\longrightarrow \hat{K}_F(S^2)$$

is the exact sequence of abelian groups. From (A) since

$$\hat{K}_F(S^3)=0\ \ (F=\boldsymbol{R}\text{ or }\boldsymbol{C})$$
$$\hat{K}_{\boldsymbol{R}}(S^2)\cong \boldsymbol{Z}/2,\ \hat{K}_{\boldsymbol{R}}(P_2(\boldsymbol{R}))\cong \boldsymbol{Z}/4$$
$$\hat{K}_{\boldsymbol{C}}(S^2)\cong \boldsymbol{Z},\ \hat{K}_{\boldsymbol{C}}(P_2(\boldsymbol{R}))\cong \boldsymbol{Z}/2$$

we have the exact sequences

$$0\longrightarrow \hat{K}_{\boldsymbol{R}}(C_f)\longrightarrow \boldsymbol{Z}/4\longrightarrow \boldsymbol{Z}/2\ \ (\text{exact})$$

and

$$0 \longrightarrow K_{\boldsymbol{C}}(C_f) \longrightarrow \boldsymbol{Z}/2 \longrightarrow \boldsymbol{Z} \text{ (exact)}.$$

Thus $\hat{K}_{\boldsymbol{R}}(C_f) \cong \hat{K}_{\boldsymbol{C}}(C_f) \cong \boldsymbol{Z}/2$. By (E) we have

$$\hat{K}_{\boldsymbol{R}}(C_f) \cong \hat{K}_{\boldsymbol{R}}(C_f') \cong \hat{K}_{\boldsymbol{R}}(p_3(\boldsymbol{R})) \cong \boldsymbol{Z}/2$$
$$\hat{K}_{\boldsymbol{C}}(C_f) \cong \hat{K}_{\boldsymbol{C}}(C_f') \cong \hat{K}_{\boldsymbol{C}}(p_3(\boldsymbol{R})) \cong \boldsymbol{Z}/2. \quad ///$$

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