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THE GROWTH FUNCTION OF TUBES ABOUT GEODESICS

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1. Introduction

Let M be an n-dimensional Riemannian manifold of class C^ω and P a submanifold, possibly with boundary. For $s>0$ let $T(P,s)$ be the set of points at distance s from P in M. We assume that s is small enough so that s is not larger than the distance between P and its nearest focal point. From now on let P be a compact orientable hypersurface. Denote by $A(s)$ the $(n-1)$-dimensional volume of the hypersurface $T(P,s)$ in the direction of a chosen normal. Following Wu and Holzsager [5], [6], we call $A(s)$ the growth function of P.

In [6], it is shown that the two-dimensional Riemannian manifolds of constant curvature equal to are characterized by the linear differential equation

$$A''(s)+cA(s)=0 \tag{1}$$

for all P. Erbacher [2] and Holzsager [5] characterized three-dimensional Riemannian manifolds of constant curvature by the linear differential equation (of lowest order)

$$A'''(s)+c_2(s)A''(s)+c_1A'(s)+c_0(s)A(s)=0 \tag{2}$$

for all P. Gray and Vanhecke [3] studied also (1) and (2) for a special class of hypersurfaces which consists of small geodesic spheres. They strengthened the results of [2], [5], [6] using the power series expansion for the volume of a geodesic sphere. The purpose of the present note is to show that the technique employing the power series expansion for the growth function allows us to prove the following more general theorems. From now on we assume $\dim M=n\geq 2$.

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THEOREM 1. *Suppose that for each short geodesic segment $\sigma \subset M$ and for all small $r > 0$, the growth function $A(s)$ of each hypersurface $T(\sigma, r)$ satisfies*

$$A''(s) + c(s)A(s) = 0, \tag{3}$$

where $c(s)$ is a function of s. Then M is a space of constant curvature of dimension 2 or 3. If $n=2$, then $c(s)=K$, where K is the sectional curvature of M: if $n=3$, then $c(s)=4K$.

THEOREM 2. *Suppose that for each short geodesic segment $\sigma \subset M$ and for all small $r > 0$, the growth function $A(s)$ of each hypersurface $T(\sigma, r)$ satisfies (2). Then M is a space of constant curvature K of dimension 2, 3, or 4. If $n=2$, then either $K=c_0(s)=0$ or $K=c_1(s)\neq 0$, $c_0(s)=Kc_2(s)$; if $n=3$, then $c_1(s)=4K$, $c_0(s)=4Kc_2(s)$; if $n=4$, then $K=c_2(s)=c_1(s)=c_0(s)=0$.*

2. Preliminaries

For the sake of completeness we recall some definitions and necessary facts. Let M be an n-dimensional Riemannian manifold with metric tensor $\langle , \rangle$. The Riemannian connection ∇ and the curvature operator R of M are given by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle \\ &\quad - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle, \\ R_{XY} &= \nabla_{[X,X]} - [\nabla_X, \nabla_Y], \end{aligned}$$

for vector fields X, Y, Z. The components of the curvature tensor will be denoted by R_{ijkl}, $1 \leq i, j, k, l \leq n$, for an orthonormal frame $\{e_1, \cdots, e_n\}$ at any point of M. Let ρ, τ, $\|\rho\|$, $\|R\|$ be the Ricci tensor, the scalar curvature tensor, the length of the Ricci tensor, and the leng.h of the curvature tensor respectively. Then

$$\rho_{\alpha\beta} = \sum_{\gamma=1}^{n} R_{\alpha\gamma\beta\gamma}, \qquad \tau = \sum_{\alpha=1}^{n} \rho_{\alpha\alpha},$$

$$\|\rho\|^2 = \sum_{\alpha,\beta=1}^{n} \rho_{\alpha\beta}^2, \qquad \|R\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{n} R_{\alpha\beta\gamma\delta}^2.$$

Let $\sigma : (a, b) \to M$ be a unit speed geodesic and $\{e_1, \cdots, e_n\}$ be an orthonormal frame field along σ such that $\dot{\sigma}(t)=e_1(\sigma(t))$. In [4] the following expression is derived for the $(n-1)$-dimensional volume $V(T(\sigma,r))$ of $T(\sigma,r)$:

$$(4)\qquad V(T(\sigma,r)) = \omega_{n-1}\int_a^b \{r^{n-2}+Cr^n+Dr^{n+2}+O(r^{n+4})\}(\sigma(t))dt,$$

where

$$C=-\frac{1}{6(n-1)}(\tau+\rho_{11})$$

and

$$D=\frac{1}{360(n+1)(n-1)}(-18\Delta\tau+5\tau^2+8\|\rho\|^2-3\|R\|^2 +33\nabla_1\nabla_1\tau-9\Delta\rho_{11}+10\tau\rho_{11}+2\sum_{i=2}^{n}\rho_{1i}^2+14\sum_{i,j=2}^{n}\rho_{ij}R_{1i1j} -6\sum_{i,j,k=2}^{n}R^2{}_{1ijk}-21\nabla_1\nabla_1\rho_{11}-3\rho_{11}^2-10\sum_{i,j=2}^{n}R_{1i1j}^2).$$

Here $\omega_{n-1}=\dfrac{2\pi^{(n-1)/2}}{\Gamma\left(\dfrac{(n-1)}{2}\right)}$ is the volume of $S^{n-1}(1)$ in $\boldsymbol{R}^n$ and $\Delta=\sum_{i=1}^{n}\nabla_i\nabla_i$ denotes the Laplacian.

Following [4] we put

$$A_m{}^x(r)=\lim_{L(\sigma)\to 0}\frac{V(T(\sigma,r))}{L(\sigma)},$$

where σ is a geodesic in M with $\sigma(0)=m$, $\dot{\sigma}(0)=x$, $x\in M_m$, $\|x\|=1$, and $L(\sigma)$ is the length of σ. We take the average $A_m(r)$ of $A_m{}^x(r)$ as x ranges over the unit sphere $S^{n-1}(1)$ in M_m. Specifically we put

$$A_m(r)=\frac{1}{\omega_{n-1}}\int_{S^{n-1}(1)}A_m{}^x(r)dx,$$

where dx is the volume element of $S^{n-1}(1)$. The power series expansion of $A_m{}^x(r)$ appears in (4) as the integrand, and the power series expansion of $A_m(r)$ is given by ([4])

(5) $$A_m(r)=\omega_{n-1}\{r^{n-2}+\bar{C}r^n+\bar{D}r^{n+2}+O(r^{n+4})\}(m),$$

where

$$\bar{C}=-\frac{n+1}{6n(n-1)}\tau$$

and

$$\bar{D}=\frac{n+3}{360n(n-1)(n+2)}\left\{-\frac{6(3n-1)}{n+3}\Delta\tau+5\tau^2+8\|\rho\|^2-3\|R\|^2\right\}.$$

For a curve σ in an n-dimensional space of constant curvature c the complete formula for $V(T(\sigma,r))$ is given by ([4])

(6) $$V(T(\sigma,r))=\omega_{n-1}\left(\frac{\sin\sqrt{c}r}{\sqrt{c}}\right)^{n-2}(\cos\sqrt{c}r)L(\sigma).$$

This fomula includes the cases $c=0$ and $c<0$ as well as $c>0$ by the usual convention: if $c=0$, we take the limit as $c\to 0$; if $c<0$, we apply the identities $\cosh\sqrt{-c}=\cos\sqrt{c}$ and $\sqrt{-1}\sinh\sqrt{-c}=\sin\sqrt{c}$.

Let $L=\frac{d}{ds}$ and c be a constant. For $n\geq 2$ we put

(7) $$L_n=\begin{cases} L^{n-1} & \text{if } c=0 \\ (L^2+c)(L^2+9c)\cdots(L^2+(n-1)^2c) & \text{if } c\neq 0,\ n \text{ is even} \\ (L^2+4c)(L^2+16c)\cdots(L^2+(n-1)^2c) & \text{if } c\neq 0,\ n \text{ is odd.}\end{cases}$$

Then we have from (6) the following proposition analogous to the one in [2] (p. 215).

PROPOSITION 3. *Suppose M is an n-dimemsional Riemannian manifold of constant curvature c. Then the growth function A of each $T(\sigma,r)$, where σ is a small geodesic segment in M and $r>0$ is sufficiently small, satisfies the linear differential equation*

(8) $$L_nA=0$$

Furthermore, this is the only linear differential equation of lowest order that A satisfies for every $T(\sigma,r)$.

3. Proof of Theorem 1

Suppose the growth function $A(s)$ satisfies (3) for each $T(\sigma,r)$.

$A_m{}^x(r+s)$ satisfies also (3) for each $x \in M_m$, $m \in M$, $\|x\|=1$. Then it is not difficult to see that the average $A_m(r+s)$ of $A_m{}^x(r+s)$ satisfies also (3) for each $m \in M$. We differentiate the power series for $A_m(r+s)$ with respect to s, put the series in (3) and set $s=0$. In this way we obtain a power series expansion in r which must be identically zero. Setting the coefficients of this power series equal to zero we obtain the following relations:

$$\begin{cases} 2 \leq n \leq 3 \\ \bar{C}n(n-1)+c(0)=0 \\ \bar{D}(n+2)(n+1)+c(0)\bar{C}=0. \end{cases}$$

If $n=2$, then we obtain $\tau(m)=2c(0)$ for all $m \in M$. Therefore it is necessary that M has constant curvature $c(0)$. On the other hand from Proposition 3 $c(s)$ must be a constant function.

If $n=3$, then we obtain $\tau=\frac{3}{2}c(0)$ and $-\frac{5}{3}\tau^2+8\|\rho\|^2-3\|R\|^2=0$. Since a three-dimensional Riemannian manifold satisfies $\|R\|^2=4\|\rho\|^2-\tau^2$, we find that $\|R\|^2=\|\rho\|^2$. According to a result of Calabi [1] this implies that M has constant curvature $\frac{1}{4}c(0)$. On the other hand from Proposition 3 $c(s)$ must be a constant function.

REMARK. We have in fact proved the following theorems. It is obvious that Theorem 4 implies Theorem 5 and Theorem 5 implies Theorem 1.

THEOREM 4. *Suppose that for all small* $r>0$ *each point* $m \in M$ *satisfies*

$$A_m''(r+s)+c(s)A_m(r+s)=0 \text{ for small } s \geq 0.$$

Then we have the same conclusion as that of Theorem 1.

THEOREM 5. *Suppose that for all small* $r>0$, *each* $x \in M_m$, $m \in M$, $\|x\|=1$ *satisfies*

$$(A_m{}^x)''(r+s)+c(s)A_m{}^x(r+s)=0 \text{ for small } s \geq 0.$$

Then we have the same conclusion as that of Theorem 1.

4. Proof of Theorem 2

We will prove the following theorem which implies Theorem 2.

THEOREM 6. *Suppose that for all small* $r>0$, *each* $x\in M_m$, $m\in M$, $\|x\|=1$ *satisfies*

$$(A_m{}^x)'''(r+s)+c_2(s)(A_m{}^x)''(r+s)+c_1(s)(A_m{}^x)'(r+s)+c_0(s)A_m{}^x(r+s)=0 \tag{9}$$

for small $s\geq 0$. *Then we have the same conclusion as that of Theorem* 2.

In the same way as in the above section we obtain the following three cases:

$$\begin{cases} n=2, \\ 2c_2(0)C+c_0(0)=0, \\ 12D+c_1(0)C=0, \\ 12c_2(0)D+c_0(0)C=0, \\ 30E+c_1(0)D=0, \end{cases} \tag{10}$$

$$\begin{cases} n=3, \\ 6C+c_1(0)=0, \\ 20D+c_1(0)C=0, \end{cases} \qquad \begin{cases} n=4, \\ c_2(0)=c_0(0)=0, \\ 24C+2c_1(0)=0, \\ 30D+c_1(0)C=0, \end{cases}$$

where C, D, E are the respective coefficients of r^n, r^{n+2}, r^{n+4} in the power series of $A_m{}^x(r+s)$.

In case of $n=0$ we need, first of all, the following expression for $V(T(\sigma, r))$, when $\sigma:(a, b)\to M$ is a unit speed curve:

$$V(T(\sigma, r))=2\int_a^b \{1+Cr^2+Dr^4+Er^6+O(r^8)\}(\sigma(t))dt, \tag{11}$$

where

$$C=-\frac{1}{2}K,$$

$$D=\frac{1}{24}(-N^2K+K^2),$$

and

$$E=\frac{1}{720}(-N^4K+7KN^2K+4(NK)^2-K^3).$$

Here K is the sectional curvature of M and N is the unit normal vector field on σ. We can derive (11) by the method of Gray and Vanhecke

[4]. We omit the tedious calculations. Now (10) and (11) give

$$(12)\qquad \begin{cases} N^2K=K^2-c_1(0)K, \\ -N^4K+7KN^2K+4(NK)^2-K^3=c_1(0)(N^2K-K^2). \end{cases}$$

Then straightforward computations show that $K=c_1(0)\neq 0$ or $K=0$. In view of Proposition 3 this leads to the conclusion when $n=2$.

If $n=3$, the assumption (9) implies (10) and, of course,

$$(10)'\qquad \begin{cases} 6\bar{C}+c_1(0)=0, \\ 20\bar{D}+c_1(0)\bar{C}=0. \end{cases}$$

According to (10)′ we have $\tau=\frac{3}{2}c_1(0)$ and $\|R\|^2=\|\rho\|^2$ for all $m\in M$. This together with Proposition 3 implies that M is a space of constant curvature K, and $c_1(s)=4K$, $c_0(s)=4Kc_2(s)$.

Finally when $n=4$, according to

$$(10)'\qquad \begin{cases} c_2(0)=c_0(0)=0, \\ 24\bar{C}+2c_1(0)=0, \\ 30\bar{D}+c_1(0)\bar{C}=0, \end{cases}$$

we obtain $\tau=\frac{6}{5}c_1(0)$ and $8\|\rho\|^2-3\|R\|^2=\frac{15}{7}\tau^2$. From (10) we have also $\rho_{ii}=-\tau+\frac{3}{2}c_1(0)$, $1\leq i\leq 4$. It follows that M is flat. Then Proposition 3 says that $c_2(s)=c_1(s)=c_0(s)=0$. This completes the proof.

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