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ON COMPLEX AND CONTACT CONFORMAL FLATNESS

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0. Introduction

K. Yano([5], [6]) defined complex conformal connections and contact conformal connections in Kaehler manifolds and Sasakian manifolds respectively and obtained the following theorems.

THEOREM A. *If, in a real n-dimensional Kaehlerian manifold* $(n\geq 4)$, *there exists a scalar function* p *such that the complex conformal connection*

$$\Gamma_{ji}{}^h=\left\{{}_j{}^h{}_i\right\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h,$$

where p_i *is the gradient of* p *and* $q_i=-p_tF_i{}^t$, *is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.*

THEOREM B. *If, in a* $(2m+1)$*-dimensional Sasakian manifold*$(m>1)$, *there exists a scalar function* p *such that the contact conformal connection*

$$\begin{aligned}\Gamma_{ji}{}^h=&\left\{{}_j{}^h{}_i\right\}+(\delta_j^h-\eta_j\eta^h)p_i+(\delta_i^h-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h\\&+F_j{}^h(q_i-\eta_i)+F_i{}^h(q_j-\eta_j)-F_{ji}(q^h-\eta^h),\end{aligned}$$

where p_i *is the gradient of* p *and* $q_i=-p_tF_i{}^t$, *is of zero curvature, then the contact Bochner curvature tensor of the manifold vanishes.*

In this paper we consider the notion of complex and contact conformal flatness respectively and obtain some results related to the converses of the above theorems.

I. Complex conformal flatness

1. Complex conformal connections

Let M be an n-dimensional Kaehler manifold covered by a system of

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coordinate neighborhoods $\{U;x^h\}$ and denote by g_{ji} and F_i^h the components of the Hermitian metric tensor and those of the complex structure tensor of M respectively. Then we have

$$(1.1)\qquad F_k^j F_i^k = -\delta_i^j,\ \ g_{kh}F_j^k F_i^h = g_{ji},\ \ \nabla_k F_i^h = 0,$$

where ∇_k is the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}. We denote by $K_{kji}{}^h$ the curvature tensor in M. It is well known that $K_{kji}{}^h$ and $K_{kjih}=K_{kji}{}^t g_{th}$ satisfy

$$(1.2)\qquad \begin{aligned} &\nabla_t K_{kji}{}^t = \nabla_k K_{ji} - \nabla_j K_{ki},\ \ K_{kjih} - K_{kjst}F_i^s F_h^t = 0,\\ &K_{jt}F_i^t + K_{it}F_j^t = 0,\ \ K_{ji} - K_{ts}F_j^t F_i^s = 0,\ \ 2\nabla_t K_j^t = \nabla_j K, \end{aligned}$$

where K_{ji} and K are the Ricci tensor and the scalar curvature of M respectively.

We now consider the so-called Bochner curvature tensor (Bochner[1], Tachibana[4]) defined by

$$(1.3)\quad \begin{aligned} B_{kji}{}^h = {} & K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k^h g_{ji} - L_j^h g_{ki} + F_k^h M_{ji} - F_j^h M_{ki}\\ & + M_k^h F_{ji} - M_j^h F_{ki} - 2(M_{kj}F_i^h + F_{kj}M_i^h), \end{aligned}$$

where

$$(1.4)\quad L_{ji} = -\frac{1}{n+4}K_{ji} + \frac{1}{2(n+2)(n+4)}Kg_{ji},\ \ M_{ji} = -L_{jt}F_i^t.$$

In a previous paper (Kim[2]), we proved that $\nabla_j K$ in a Kaehler manifold M with vanishing Bochner curvature tensor is contravariant analytic.

Let us consider a conformal change of Hermitian metric

$$\bar{g}_{ji} = e^{2p}g_{ji},\ \ \bar{F}_i^h = F_i^h,\ \ \bar{F}_{ji} = e^{2p}F_{ji},$$

where p is a scalar function $(n \geq 4)$.

The affine connection Γ_{ji}^h which satisfies

$$D_k\bar{g}_{ji} = 0,\ \ D_k\bar{F}_i^h = 0 \text{ and } \Gamma_{ji}^h - \Gamma_{ij}^h = -2F_{ji}q^h,$$

where q^h is a vector field and D_k is the operator of covariant differentiation with respect to Γ_{ji}^h, is given by

$$(1.5)\qquad \Gamma_{ji}^h = \{{}_j{}^h{}_i\} + \delta_j^h p_i + \delta_i^h p_j - g_{ji}p^h + F_j^h q_i + F_i^h q_j - F_{ji}q^h,$$

where $p_i = \frac{\partial}{\partial x^i} p$, $p^h = p_t g^{th}$, $q_i = -p_t F_i^t$, $q^h = q_t g^{th}$.

K. Yano([5]) called such an affine connection a complex conformal connection. The curvature tensor $R_{kji}{}^h$ of Γ_{ji}^h is given by

$$(1.6)\quad R_{kji}{}^h = K_{kji}{}^h - \delta_k^h P_{ji} + \delta_j^h P_{ki} - P_k{}^h g_{ji} + P_j{}^h g_{kj} - F_k{}^h Q_{ji} + F_j{}^h Q_{ki} - Q_k{}^h F_{ji} + Q_j{}^h F_{ki} + (\nabla_k q_j - \nabla_j q_k) F_i{}^h - 2F_{kj}(p_i q^h - q_i p^h),$$

where

$$(1.7)\quad P_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} p_t p^t g_{ji},\quad Q_{ji} = \nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2} p_t p^t F_{ji}$$

and consequently

$$(1.8)\qquad Q_{ji} = -P_{jt} F_i^t,\quad P_{ji} = Q_{jt} F_i^t.$$

If there exists in M a scalar function p such that the curvature tensor $R_{kji}{}^h$ vanishes, then the Kaehler manifold M with the metric tensor g_{ji} is said to be complex conformally flat.

In a previous paper([2]), the present author proved the following theorems.

THEOREM C. *Let M be an n-dimensional Kaehler manifold $(n \geq 4)$. Then a necessary and sufficient condition that the curvature tensor of a complex conformal connection* (1.5) *coincides with the Bochner curvature tensor of M is that there exists a scalar function p such that*

$$(1.9)\qquad K_{jk} = -(n+4)(p_t p^t g_{jk} + p_j p_k + q_j q_k),$$

$$(1.10)\qquad \nabla_j p_k = -p_t p^t g_{jk} - 2q_j q_k.$$

THEOREM D. *If M is complex conformally flat, then there exists a scalar function p which satisfies* (1.9) *and* (1.10).

We can see easily that (1.10) is equivalent to

$$(1.11)\qquad \nabla_j q_k = -p_t p^t g_{jk} + 2q_j p_k.$$

Now, suppose that there exists a scalar function p which satisfies (1.9) and (1.10). Differentiating (1.9) covariantly along M, we have

$$\nabla_i K_{jk} = -(n+4)(2g_{jk} p^t \nabla_i p_t + p_k \nabla_i p_j + p_j \nabla_i p_k + q_k \nabla_i q_j + q_j \nabla_i q_k).$$

Calculating $(\nabla_i K_{jk} - \nabla_j K_{ik}) g^{ik}$ and taking account of $\nabla_t q^t = 0$, we can find

$$(1.12)\qquad \nabla_j K=2(n+4)\{-(2n+1)p^t\nabla_j p_t+p_j\nabla_t p^t+q^t\nabla_t q_j\}.$$

From (1.9), (1.10) and (1.11) we have

$$(1.13)\qquad \begin{aligned}&\nabla_t p^t=-(n+2)p_t p^t,\quad q^t\nabla_t q_j=3p_t p^t p_j,\\ &p^t\nabla_t p_j=-p_t p^t p_j,\quad K=-(n+2)(n+4)p_t p^t,\end{aligned}$$

which and (1.12) imply

$$(1.14)\qquad \nabla_j K=-2Kp_j.$$

From (1.13) we can see that $K\leq 0$. Moreover, we assume that the scalar curvature K is constant. Then we have $K=0$ by the help of (1.13) and (1.14). In this case we have $p_j=0$. Thus we have the following, by the help of (1.6),

THEOREM 1.1. *Let M be an n-dimensional Kaehler manifold ($n\geqq 4$) with constant scalar curvature. If M is complex conformally flat, then M is flat.*

2. Kaehler manifolds with nonconstant negative scalar curvature

Let M be a Kaehler manifold with nonconstant negative scalar curvature. For the scalar function $p=-\frac{1}{2}\log(-K)$, we consider the complex conformal connection (1.5). In this case, we have

$$(2.1)\qquad p_j=-\frac{1}{2K}\nabla_j K,\quad \nabla_k\nabla_j K=4Kp_k p_j-2K\nabla_k p_j.$$

In this section, we assume that the Ricci tensor of M satisfies

$$(2.2)\qquad K_{ji}=-(n+4)(\lambda g_{ji}+p_j p_i+q_j q_i),$$

where we have put $\lambda=p_t p^t$. Then we have the following equations:

$$(2.3)\qquad \begin{aligned}&K=-(n+2)(n+4)\lambda,\quad K_{jt}p^t=-2(n+4)\lambda p_j,\\ &\nabla_j\lambda=-2\lambda p_j=\frac{2K}{(n+2)(n+4)}p_j=2p^t\nabla_j p_t,\quad \lambda^2=-p^t p^s\nabla_s p_t.\end{aligned}$$

Now we suppose that $\nabla_j K$ is a contravariant analytic vector field. Then we have

(2.4) $$\nabla_i q_j + \nabla_j q_i = 2(q_i p_j + q_j p_i).$$

Differentiating (2.2) covariantly, we have

(2.5) $$\nabla_k K_{ji} = -(n+4)(g_{ji}\nabla_k\lambda + p_i\nabla_k p_j + p_j\nabla_k p_i + q_i\nabla_k q_j + q_j\nabla_k q_i).$$

Contracting (2.5) with g^{ki} and by the help of (2.1), (2.3) and (2.4), we have

(2.6) $$\nabla_t p^t = -(n+2)\lambda = \frac{K}{n+4}.$$

From (2.3) and (2.6), we can find

(2.7) $$\nabla^t\nabla_t\lambda = 2(n+4)\lambda^2 = 2p^k\nabla^t\nabla_t p_k + 2\nabla_s p_t\nabla^s p^t.$$

Contracting the Ricci identity $\nabla_k\nabla_j p_t = \nabla_j\nabla_k p_t - K_{kjt}{}^s p_s$ with g^{kt}, we have

$$\nabla^t\nabla_t p_j = \nabla_j(\nabla_t p^t) + K_j{}^t p_t,$$

which, by the help of (2.3) and (2.6), implies

(2.8) $$(\nabla^t\nabla_t p_j)p^j = -4\lambda^2.$$

From (2.7) and (2.8), we find

(2.9) $$\nabla_s p_t \nabla^s p^t = (n+8)\lambda^2.$$

Transvecting (2.4) with $q^i p^j$ and using (2.3) and $p_t q^t = 0$, we have

(2.10) $$(\nabla_i p_j)q^i q^j = -3\lambda^2.$$

Since

$$\|\nabla_i p_k + \lambda g_{ik} + 2q_i q_k\|^2 = (\nabla_i p_k)(\nabla^i p^k) + 2\lambda\nabla_t p^t + 4(\nabla_i p_k)q^i q^k + (n+8)\lambda^2,$$

we have, by the help of (2.6), (2.9) and (2.10),

$$\nabla_i p_k + \lambda g_{ik} + 2q_i q_k = 0.$$

Thus we have, by the help of the theorem C, the following

THEOREM 2.1. *Let M be a Kaehler manifold with nonconstant negative scalar curvature and let* $p = -\frac{1}{2}\log(-K)$. *If the Ricci tensor of M satisfies* (1.9) *and $\nabla_j K$ is contravariant analytic, then the curvature tensor* *of the complex conformal connection* (1.5) *coincides with the Bochner curvature tensor of M.*

If M is a Kaehler manifold with vanishing Bochner curvature tensor, then $\nabla_j K$ is a contravariant analytic vector. Hence we have, by the help of theorem 2.1, the following

THEOREM 2.2. *Let M be a Kaehler manifold with nonconstant negative scalar curvature and with vanishing Bochner curvature tensor. If the Ricci tensor of M satisfies*

$$K_{ji} = -(n+4)(p_t p^t g_{ji} + p_j p_i + F_j{}^t F_i{}^s p_t p_s), \tag{1.9}$$

where $p_j = \partial_j p$ *and* $p = -\frac{1}{2}\log(-K)$, *then M is complex conformally flat.*

II. Contact conformal flatness

3. Contact conformal connections

Let M be a $(2m+1)$-dimensional dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field $F_i{}^h$ of type (1, 1), a vector field ξ^h and a 1-form η_i satisfying

$$F_j{}^i F_i{}^h = -\delta_j^h + \eta_j \xi^h, \quad F_i{}^h \xi^i = 0, \quad \eta_i F_j{}^i = 0, \quad \eta_i \xi^i = 1, \tag{3.1}$$

where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m+1\}$. Such a set of a tensor field F of type (1, 1), a vector field ξ and a 1-form η is called an almost contact structure and a manifold with an almost contact structure an almost contact manifold. (Yano[6]).

If the set (F, ξ, η) satisfies

$$N_{ji}{}^h + (\partial_j \eta_i - \partial_i \eta_j) = 0,$$

where $N_{ji}{}^h$ is the Nijenhuis tensor formed with $F_i{}^h$, then the almost contact structure is said to be normal and the manifold is called a normal almost contact manifold. If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$g_{ts} F_j{}^t F_i{}^s = g_{ji} - \eta_j \eta_i, \quad \eta_i = g_{ih} \xi^h, \tag{3.2}$$

then the almost contact structure is said to be metric and the manifold is called an almost contact metric manifold. We shall write η^h instead of ξ^h in the sequel. If an almost contact metric manifold satisfies $F_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j)$, then the almost contact metric structure is called a contact structure. A manifold with a normal contact structure is called a Sasakian manifold. It is well known that in a Sasakian manifold, we have

$$(3.3)\qquad \nabla_j\eta^h=F_i^{\ h},\quad \nabla_jF_i^{\ h}=-g_{ji}\eta^h+\delta_j^h\eta_i,\quad F_{ji}=-F_{ij},$$

$$(3.4)\quad K_{kji}{}^t\eta_t=\eta_k g_{ji}-\eta_j g_{ki},\quad K_{jt}\eta^t=2m\eta_j,\quad \eta^t\nabla_tK_{ji}=0,\quad \eta^t\nabla_tK=0,$$

$$(3.5)\qquad K_{jt}F_i^{\ t}+K_{it}F_j^{\ t}=0,$$

where $K_{kji}{}^h$, K_{ji} and K are the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

The contact Bochner curvature tensor (Yano[6]) is defined by

$$\begin{aligned}(3.6)\quad B_{kji}{}^h=&K_{kji}{}^h+(\delta_k^h-\eta_j\eta^h)L_{ki}+L_k^{\ h}(g_{ji}-\eta_j\eta_i)-L_j^{\ h}(g_{ki}-\eta_k\eta_i)\\&+F_k^{\ h}M_{ji}-F_j^{\ h}M_{ki}+M_k^{\ h}F_{ji}-M_j^{\ h}F_{ki}\\&-2(M_{kj}F_i^{\ h}+F_{kj}M_i^{\ h})+(F_k^{\ h}F_{ji}-F_j^{\ h}F_{ki}-2F_{kj}F_i^{\ h}),\end{aligned}$$

where

$$(3.7)\quad L_{ji}=-\frac{1}{2(m+2)}[K_{ji}+(L+3)g_{ji}-(L-1)\eta_j\eta_i],\quad L_k^{\ h}=L_{kt}g^{th},$$

$$(3.8)\quad M_{ji}=-L_{jt}F_i^{\ t},\quad M_k^{\ h}=M_{kt}g^{th},\quad L=g^{ji}L_{ji}=-\frac{K+2(3m+2)}{4(m+1)}.$$

From (3.7) and (3.8), using (3.4), we find

$$(3.9)\qquad L_{jt}\eta^t=-\eta_j,\quad M_{ji}\eta^i=0,\quad M_{jt}F_i^{\ t}=L_{ji}+\eta_j\eta_i.$$

In a Sasakian manifold with structure tensor $(F_i^{\ h},\ \eta_i,\ g_{ji})$, the affine connection D which satisfies

$$D_k(e^{2p}g_{ji})=2e^{2p}p_kp_j\eta_j\eta_i,\quad D_jF_i^{\ h}=0,\quad D_j\eta^h=0$$

and whose torsion tensor satisfies

$$\Gamma_{ji}^{\ h}-\Gamma_{ij}^{\ h}=-2F_{ji}u^h,$$

where p is a scalar function and u^h a vector field, is given by

$$\begin{aligned}(3.10)\quad \Gamma_{ji}^{\ h}=&\left\{{h \atop j\ i}\right\}+(\delta_j^h-\eta_j\eta^h)p_i+(\delta_i^h-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h\\&+F_j^{\ h}(q_i-\eta_i)+F_i^{\ h}(q_j-\eta_j)-F_{ji}(q^h-\eta^h),\end{aligned}$$

where

$$(3.11)\qquad p_i=\partial_i p,\quad p^h=p_t g^{th},\quad q_i=-p_t F_i^{\ t},\quad q^h=q_t g^{th}$$

and p satisfies $p_i\eta^i=0$.

K. Yano([6]) called such an affine connection a contact conformal connection. From (3.11) and $p_i\eta^i=0$ we see that

$$(3.12)\qquad q_t F_i^{\ t}=p_i,\quad F_t^{\ h}p^t=q^h,\quad F_t^{\ h}q^t=-p^h,\quad p_i\eta^i=0,$$
$$q_i\eta^i=0,\quad p_t q^t=0,\quad p^t p_t=q^t q_t.$$

The curvature tensor $R_{kji}^{\ \ \ h}$ of $\Gamma_{ji}^{\ h}$ is given by

$$\begin{aligned}(3.13)\qquad R_{kji}^{\ \ \ h}=&K_{kji}^{\ \ \ h}-(\delta_k^h-\eta_k\eta^h)p_{ji}+(\delta_j^h-\eta_j\eta^h)p_{ki}-p_k^{\ h}(g_{ji}-\eta_j\eta_i)\\&+p_j^{\ h}(g_{ki}-\eta_k\eta_i)-F_k^{\ h}q_{ji}+F_j^{\ h}q_{ki}-q_k^{\ h}F_{ji}+q_j^{\ h}F_{ki}\\&+(\nabla_k q_j-\nabla_j q_k)F_i^{\ h}+2F_{kj}(q_i p^h-p_i q^h)\\&+(F_k^{\ h}F_{ji}-F_j^{\ h}F_{ki}-2F_{kj}F_i^{\ h}),\end{aligned}$$

where

$$(3.14)\qquad p_{ji}=\nabla_j p_i-p_j p_i+(q_j-\eta_j)(q_i-\eta_i)+\frac{1}{2}p_t p^t(g_{ji}-\eta_j\eta_i),$$

$$(3.15)\qquad q_{ji}=\nabla_j q_i-p_j(q_i-\eta_i)-p_i(q_j-\eta_j)+\frac{1}{2}p_t p^t F_{ji}.$$

From (3.12), (3.14), (3.15) and $p_i\eta^i=0$, we find

$$(3.16)\qquad p_{ji}=p_{ij},\quad \eta^j p_{ji}=\eta_i,\quad q_{ji}=-p_{jt}F_i^{\ t},\quad \eta^j q_{ji}=0,\quad q_{ji}\eta^i=0,$$
$$p_{ji}=q_{jt}F_i^{\ t}+\eta_j\eta_i,\quad q_{ts}F_j^{\ t}F_i^{\ s}=-q_{ij}.$$

If, in a $(2m+1)$-dimensional Sasakian manifold $(2m+1>3)$, there exists a scalar function p such that the contact conformal connection (3.10) is of zero curvature, then we call such a manifold contact conformally flat one. From theorem B, we can see that if M is contact conformally flat, then it is a manifold with vanishing contact Bochner curvature tensor.

4. Contact Bochner curvature tensor and curvature tensor of contact conformal connection

We now assume that there exists a scalar function p such that $R_{kji}^{\ \ \ h}=B_{kji}^{\ \ \ h}$. Then we have

$$(4.1)\quad \begin{aligned}&(g_{kh}-\eta_k\eta_h)(p_{ji}+L_{ji})-(g_{jh}-\eta_j\eta_h)(p_{ki}+L_{ki})\\&+(p_{kh}+L_{kh})(g_{ji}-\eta_j\eta_i)-(p_{jh}+L_{jh})(g_{ki}-\eta_k\eta_i)\\&+F_{kh}(q_{ji}+M_{ji})-F_{jh}(q_{ki}+M_{ki})+(q_{kh}+M_{kh})F_{ji}\\&-(q_{jh}+M_{jh})F_{ki}+(A_{kj}-2M_{kj})F_{ih}\\&+F_{kj}(B_{ih}-2M_{ih})=0,\end{aligned}$$

where

$$(4.2)\qquad A_{kj}=-(\nabla_k q_j-\nabla_j q_k),$$

$$(4.3)\qquad B_{kj}=2(p_k q_j-q_k p_j),$$

and consequently,

$$(4.4)\quad A=F^{kj}A_{kj}=-2\nabla_t p^t,\quad B=F^{kj}B_{kj}=4p_t p^t,\quad \eta^k A_{kj}=0,\quad \eta^k B_{kj}=0.$$

Interchanging k with i and h with j in (4.1) respectively, subtracting the resulting equation from (4.1) and taking account of $p_{ji}=p_{ij}$, $L_{ji}=L_{ij}$ and $M_{ji}=-M_{ij}$, we obtain

$$(4.5)\quad \begin{aligned}&F_{kh}(q_{ji}+q_{ij})-F_{jh}(q_{ki}+q_{ik})+(q_{kh}+q_{hk})F_{ji}-(q_{jh}+q_{hj})F_{ki}\\&+(A_{kj}-B_{kj})F_{ih}-F_{kj}(A_{ih}-B_{ih})=0.\end{aligned}$$

Transvecting (4.5) with F^{kh}, we find, by the help of (4.4),

$$(4.6)\qquad q_{ij}+q_{ji}=0,$$

which and (3.16) imply

$$(4.7)\qquad q_{ji}=q_{ts}F_j{}^t F_i{}^s.$$

Substituting (4.6) into (4.5), we find

$$(4.8)\qquad (A_{kj}-B_{kj})F_{ih}-F_{kj}(A_{ih}-B_{ih})=0,$$

from which, by transvection with F^{kj}

$$A_{ih}-B_{ih}=\frac{1}{2m}(A-B)F_{ih}$$

and consequently, using (4.4),

$$(4.9)\qquad A_{ih}-B_{ih}=-\frac{1}{m}(\nabla_t p^t+2p_t p^t)F_{ih}.$$

On the other hand, from the definitions of q_{ji} and A_{ji}, we find

$$(4.10)\qquad A_{ji}=-2q_{ji}+p_tp^tF_{ji}.$$

Since

$$F^{kj}q_{kj}=F^{kj}(-p_{kt}F_j^{\ t})=p_t^{\ t}-1,$$

we have

$$(4.11)\qquad A=-2(p_t^{\ t}-1)+2mp_tp^t.$$

From (4.4) and (4.11), we obtain

$$(4.12)\qquad p_t^{\ t}=\nabla_tp^t+mp_tp^t+1.$$

Equations (4.9) and (4.10) give

$$B_{ji}=-2q_{ji}+\frac{1}{m}\{\nabla_tp^t+(m+2)p_tp^t\}F_{ji},$$

which and (4.12) imply

$$(4.13)\qquad B_{ji}=-2q_{ji}+\frac{1}{m}\{p_t^{\ t}+2p_tp^t-1\}F_{ji}.$$

By the cyclic sum of (4.1) with respect to the indices k, j and i and using $p_{ji}=p_{ij}$, $M_{ji}=-M_{ij}$ and $q_{ij}=-q_{ji}$, we have

$$(4.14)\qquad \begin{aligned}&F_{kh}(2q_{ji}+A_{ji})+F_{jh}(2q_{ih}+A_{ik})+F_{ih}(2q_{kj}+A_{kj})\\ &+F_{ji}(2q_{kh}+B_{kh})+F_{ik}(2q_{jh}+B_{jh})+F_{kj}(2q_{ih}+B_{ih})=0.\end{aligned}$$

Substituting (4.10) and (4.13) into (4.14), we find

$$(4.15)\qquad p_t^{\ t}+(m+2)p_tp^t-1=0.$$

Thus equation (4.13) can be written as

$$(4.16)\qquad B_{ji}=-2q_{ji}-p_tp^tF_{ji}.$$

Transvecting (4.1) with g^{kh} and using (4.10) and (4.13), we find

$$(4.17)\qquad (2m+5)(p_{ji}+L_{ji})+(p_t^{\ t}+L_t^{\ t})g_{ji}=0.$$

Transvecting (4.17) with g^{ji} and using (3.8), we have

$$(4.18)\qquad p_t^{\ t}=-L_t^{\ t}=\frac{K+2(3m+2)}{4(m+1)},$$

and consequently, using (4.17),

(4.19) $$L_{ji}=-p_{ji},$$

which and (4.15) imply

(4.20) $$p_t p^t=-\frac{2m+K}{4(m+1)(m+2)}.$$

Since $q_{ji}=-p_{jt}F_i^t$ and $p_{jt}=-L_{jt}$, we obtain

(4.21) $$q_{ji}=-M_{ji}$$

Substituting (4.19) and (4.21) into (4.1) and taking account of (4.2) and (4.3), we have

(4.22) $$(\nabla_k q_j-\nabla_j q_k+2M_{kj})F_{ih}=2F_{kj}(p_i q_h-q_i p_h-M_{ih}).$$

Since $\nabla_k q_j-\nabla_j q_k=2q_{kj}-p_t p^t F_{kj}$, we have

$$-p_t p^t F_{kj}F_{ih}=2F_{kj}(p_i q_h-q_i p_h-M_{ih}),$$

and consequently

$$q_{ih}=-p_i q_h+q_i p_h-\frac{1}{2}p_t p^t F_{ih}.$$

Comparing the last equation with (3.15), we have

(4.23) $$\nabla_i q_h+p_t p^t F_{ih}-2q_i p_h+p_i\eta_h+p_h\eta_i=0.$$

Conversely, we now assume that there exists a scalar function p in M which satisfies (4.19) and (4.23). Then we have

(4.24) $$R_{kji}{}^h=B_{kji}{}^h+W_{kji}{}^h,$$

where

(4.25) $$W_{kji}{}^h=2(M_{kj}F_i{}^h+F_{kj}M_i{}^h)+(\nabla_k q_j-\nabla_j q_k)F_i{}^h+2F_{kj}(q_i p^h-p_i q^h),$$

Since $\nabla_k q_j-\nabla_j q_k=2q_{kj}-p_t p^t F_{kj}$, (4.25) can be rewritten as

(4.26) $$W_{kji}{}^h=2F_{kj}\left(-q_i{}^h-\frac{1}{2}p_t p^t F_i{}^h+q_i p^h-p_i q^h\right).$$

Substituting (3.15) into (4.26) and taking account of (4.23), we find $W_{kji}{}^h=0$, which and (4.24) give $R_{kji}{}^h=B_{kji}{}^h$. Thus we have the following

THEOREM 4.1. *Let M be a $(2m+1)$-dimensional Sasakian manifold $(m>1)$. Then a necessary and sufficient condition that the curvature tensor of a contact conformal connection* (3.10) *coincides with the contact Bochner curvature tensor of M is that there exists a scalar function p which satisfies* (4.19) *and* (4.23).

We can easily see that (4.23) is equivalent to

$$(4.27)\qquad \nabla_i p_j = -p_t p^t (g_{ji}-\eta_i\eta_j) - 2q_i q_j + q_j\eta_i + q_i\eta_j.$$

Now, we assume that there exists a scalar fuuction p which satisfies $L_{ji}=-p_{ji}$ and (4.27). Then we have

$$\begin{aligned} K_{ji} = &-(L+3)g_{ji} + (L-1)\eta_j\eta_i \\ &+2(m+2)\left\{-\frac{1}{2}p_t p^t (g_{ji}-\eta_j\eta_i) - q_j q_i - p_j p_i + \eta_j\eta_i\right\}, \end{aligned}$$

which and (4.20) imply

$$(4.28)\quad K_{ji}-2m\eta_j\eta_i = -2(m+2)\left\{\left(p_t p^t + \frac{1}{m}\right) + 2(g_{ji}-\eta_j\eta_i) + q_j q_i + p_j p_i\right\},$$

or equivalently

$$\begin{aligned} (4.29)\qquad & K_{ji}-2m\eta_j\eta_i \\ &= -2(m+2)\left\{\frac{2m+2-K}{4(m+1)(m+2)}(g_{ji}-\eta_j\eta_i) + p_j p_i + q_j q_i\right\}. \end{aligned}$$

Conversely, suppose that there exists a scalar function p which satisfies (4.27) and (4.28). Then we have

$$p_{ji} = -\frac{1}{2}p_t p^t (g_{ji}-\eta_j\eta_i) - q_j q_i - p_j p_i + \eta_j\eta_i.$$

From the assumption (4.28), we have

$$p_t p^t = -\frac{K+2m}{4(m+1)(m+2)} = \frac{L+1}{m+2},$$

which gives $p_{ji}=-L_{ji}$. Thus we have

THEOREM 4.2. *Let M be a $(2m+1)$-dimensional Sasakian manifold $(m>1)$. Then a necessary and sufficient condition that the curvature tensor* *of a contact conformal connection* (3.10) *coincides with the contact Bochner curvature tensor of M is that there exists a scalar function p which satisfies* (4.27) *and* (4.28).

Combining theorem 4.2 and theorem B, we have the following

COROLLARY 4.3. *If, in a* $(2m+1)$*-dimensional Sasakian manifold* $(m>1)$, *there exists a scalar function p such that the contact conformal connection* (3.10) *is of zero curvature, then p satisfies* (4.27) *and* (4.28).

Now suppose that there exists a scalar function p such that (4.27) (or equivalently (4.23)) and (4.28) hold. Differentiating (4.28) covariantly along M, we have

$$(4.30)\quad \begin{aligned} &\nabla_i K_{jk} - 2mF_{ij}\eta_k - 2m\eta_j F_{ik} \\ &= -2(m+2)\Big\{2(g_{jk}-\eta_j\eta_k)p^t\nabla_i p_t + \Big(p_t p^t + \frac{1}{m+2}\Big)(-F_{ij}\eta_k - \eta_j F_{ik}) \\ &+ q_k\nabla_i q_j + q_j\nabla_i q_k + p_k\nabla_i p_j + p_j\nabla_i p_k\Big\}. \end{aligned}$$

Transvecting (4.30) with g^{ik} and taking account of (4.23), (4.27) and

$$(4.31)\qquad p_t p^t = -\frac{K+2m}{4(m+1)(m+2)},$$

we obtain

$$(4.32)\qquad \nabla_j K = -2(K+2m)p_j.$$

Moreover we assume that $K+2m$ never vanishes. Then we can see, by the help of (4.31), that $K+2m$ is negative nonconstant. In this case we have

$$(4.33)\qquad p_j = -\frac{1}{2(K+2m)}\nabla_j K,$$

which implies

$$(4.34)\qquad p = -\frac{1}{2}\log(-K-2m) + c,$$

where c is a constant.

Thus we have the following

THEOREM 4.4. *Let M be a* $(2m+1)$*-dimensional Sasakian manifold*

$(m>1)$. If there exists a scalar function p such that the curvature tensor of a contact conformal connection (3. 10) *coincides with the contact Bochner curvature tensor of M, then p satisfies* (4. 32). *Moreover, if $K+2m$ never vanishes, then p satisfies* (4. 34).

COROLLARY 4. 5. *Let M be a $(2m+1)$-dimensional Sasakian manifold $(m>1)$. If there exists a scalar function p such that the curvature tensor of a contact conformal connection* (3. 10) *is of zero curvature, that is, M is contact conformally flat, then p satisfies* (4. 32). *Moreover, if $K+2m$ never vanishes, then p satisfies* (4. 34).

5. Sasakian manifolds with constant scalar curvatures

In this section we characterize contact conformally flat manifolds with constant scalar curvatures. Suppose that there exists a scalar function p such that the curvature tensor of contact conformal connection (3. 10) is zero, that is, M is contact conformally flat. Moreover, we assume that the scalar curvature of the manifold is constant. Then we have, by the help of corollary 4. 5,

$$(K+2m)p_j=0. \tag{5.1}$$

Transvecting (5. 1) with p^j and taking account of (4. 31), we obtain $K=-2m$. Consequently, we have $p_i=0$ and hence p is constant. In this case we have the following.

$$\Gamma_{ji}{}^h=\left\{{}_j{}^h{}_i\right\}-F_j{}^h\eta_i-F_i{}^h\eta_j+F_{ji}\eta^h,\quad D_k g_{ji}=0,\quad D_j F_i{}^h=0,\quad D_j\eta^h=0. \tag{5.2}$$

Substituting $p_i=0$ into (4. 28), we have

$$K_{ji}=-2g_{ji}+2(m+1)\eta_j\eta_i, \tag{5.3}$$

that is, the manifold M is C-Einstein.

Here we refer the following theorem

THEOREM E(M. Matsumoto and G. Chūman[3]). *The contact Bochner curvature tensor coincides with $U_{abc}{}^d$ if and only if M is a C-Einstein space, where*

$$\begin{aligned}U_{abc}{}^d=K_{abc}{}^d-(\rho+1)(g_{bc}\delta_a{}^d-g_{ac}\delta_b{}^d)-\rho(&g_{ac}\eta_b\eta^d+\eta_a\eta_c\delta_b{}^d-g_{bc}\eta_a\eta^d\\ &-\eta_b\eta_c\delta_a{}^d+F_{bc}F_a{}^d-F_{ac}F_b{}^d-2F_{ab}F_c{}^d)\end{aligned}$$

and $\rho+1=\frac{k}{2m}$, $k=\frac{K+2m}{2(m+1)}$.

Thus we have, by the help of corollary 4.3, theorem B, theorem E and (5.3), the following theorem.

THEOREM 5.1. *Let M be a $(2m+1)$-dimensional Sasakian manifold $(m>1)$ with constant scalar curvature. If M is contact conformally flat, then M is a Sasakian space form $M(-3)$.*

Now, we suppose that M is a Sasakian space form $M(-3)$. Then M is a manifold of vanishing contact Bochner curvature tensor and the scalar curvature of M is constant. The Ricci tensor of M is given by

$$K_{ji}=-2g_{ji}+(2m+2)\eta_j\eta_i. \tag{5.4}$$

We choose arbitrary constant p and consider a contact conformal connection (3.10). In this case, since $p_j=0$, the contact conformal connection is given by (5.2) and the equations (4.27) and (4.28) are satisfied. Hence $B_{kji}{}^h=R_{kji}{}^h$ by the help of theorem 4.2. Since $B_{kji}{}^h=0$, we have $R_{kji}{}^h=0$. Thus we have

THEOREM 5.2. *If M is a $(2m+1)$-dimensional Sasakian space form $M(-3)$, then M is contact conformally flat $(m>1)$.*

6. A sufficient condition for M to be contact conformally flat

In this section, we assume that the contact Bochner curvature tensor of M vanishes and $K+2m$ is negative nonconstant. Let $p=-\frac{1}{2}\log(-K-2m)+c$. Then, since $\eta^j p_j=0$, we can consider the contact conformal connection (3.10). In this case, we have

$$p_j=-\frac{1}{2(K+2m)}\nabla_j K, \tag{6.1}$$

$$\nabla_j K=-2(K+2m)p_j. \tag{6.2}$$

Differentiating (6.2) covariantly along M, we have

$$\nabla_j\nabla_j K=2(K+2m)(2p_k p_j-\nabla_k p_j). \tag{6.3}$$

THEOREM F (M. Matsumoto and G. Chūman[3]). *In a Sasakian space M(dim $M\geq 5$) with vanishing contact Bochner curvature tensor, $\nabla_j K$ is*

C-analytic.

By theorem F, we have

$$\nabla_k \nabla_j K = F_k{}^r F_j{}^s \nabla_r \nabla_s K + (\nabla^r K)(F_{rk}\eta_j + F_{rj}\eta_k). \tag{6.4}$$

Substituting (6.2) and (6.3) into (6.4), we obtain

$$2p_k p_j - \nabla_k p_j = 2q_k q_j - F_k{}^r F_j{}^s \nabla_r p_s - (q_k \eta_j + q_j \eta_k), \tag{6.5}$$

from which we find

$$\eta^r \eta^s \nabla_r p_s = 0 \tag{6.6}$$

and

$$2p_k p_j - \nabla_k p_j = 2q_k q_j + F_k{}^r \nabla_r q_j - 2q_k \eta_j - q_j \eta_k. \tag{6.7}$$

Now, suppose that the Ricci tensor of M satisfies

$$\begin{aligned} K_{ji} - 2m\eta_j\eta_i & \\ &= -2(m+2)\left\{\frac{2m+2-K}{4(m+1)(m+2)}(g_{ji} - \eta_j\eta_i) + p_j p_i + q_j q_i\right\}. \end{aligned} \tag{6.8}$$

Then we have

$$p_t p^t = -\frac{K+2m}{4(m+1)(m+2)}, \tag{6.9}$$

which and (6.2) imply

$$p^t \nabla_k p_t = \frac{K+2m}{4(m+1)(m+2)} p_k. \tag{6.10}$$

If we put $S_{ij} = F_i{}^t K_{tj}$, then we have

$$S_{ij} = -2(m+2)\left\{\frac{2m+2-K}{4(m+1)(m+2)} F_{ij} - q_i p_j + p_i q_j\right\}. \tag{6.11}$$

Differentiating (6.11) covariantly along M, we get

$$\begin{aligned} \nabla_k S_{ij} = -2(m+2)\Big\{&\frac{2(K+2m)}{4(m+1)(m+2)} p_k F_{ij} \\ &+ \frac{2m+2-K}{4(m+1)(m+2)}(\eta_i g_{kj} - \eta_j g_{ki}) \\ &- (\nabla_k q_i) p_j - q_i \nabla_k p_j + (\nabla_k p_i) q_j + p_i \nabla_k q_j\Big\}. \end{aligned} \tag{6.12}$$

Transvecting (6. 7) with q^j and taking account of (6. 10), we have

$$(6.13) \qquad (\nabla_k p_t) q^t = \frac{K+2m}{4(m+1)(m+2)} (3q_k - \eta_k).$$

In a Sasakian manifold with vanishing contact Bochner curvature tensor, we can obtain the following (See[3])

$$(6.14) \quad \nabla_k S_{ij} = \eta_i K_{jk} - \eta_j K_{ik} + \frac{1}{4(m+1)} \{F_{ik}\delta_j^r - F_{jk}\delta_i^r + 2F_{ij}\delta_k^r + (g_{jk} - \eta_j\eta_k) F_i^r - (g_{ik} - \eta_i\eta_k) F_j^r\} \nabla_r K.$$

From (6. 12), (6. 14) and (6. 2), we find

$$(6.15) \quad \frac{2(K+2m)}{4(m+1)(m+2)} p_k F_{ij} + \frac{2m+2-K}{4(m+1)(m+2)} (\eta_i g_{kj} - \eta_j g_{ki}) - (\nabla_k q_i) p_j - q_i \nabla_k p_j + q_j \nabla_k p_i + p_i \nabla_k q_j = -\frac{1}{2(m+2)} (\eta_i K_{jk} - \eta_j K_{ik}) + \frac{(K+2m)}{4(m+1)(m+2)} \{F_{ik} p_j - F_{jk} p_i + 2F_{ij} p_k - (g_{jk} - \eta_j\eta_k) q_i + (g_{ik} - \eta_i\eta_k) q_j\}.$$

Transvecting (6. 15) with q^j and taking account of (6. 13), we have

$$(6.16) \quad \nabla_k p_i = \frac{K+2\mathrm{m}}{4(m+1)(m+2)} (g_{ki} - \eta_k\eta_i) - 2q_k q_i + \eta_k q_i + \eta_i q_k.$$

Thus we have the following

THEOREM 6. 1. *Let M be a $(2m+1)$-dimensional Sasakian manifold with vanishing contact Bochner curvature tensor and let $K+2m$ be nonconstant negative $(m>1)$. If the Ricci tensor of M satisfies* (6. 8), *then M is contact conformally flat.*

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