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A CONSTRUCTIVE PROOF OF THE EXISTENCE OF GREEN'S FUNCTION ON MANIFOLDS

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1. Introduction

Let M be a complete Riemannian manifold and Δ be the Laplace operator acting on C^∞ functions on M. Then the Green's function on M is a function on $M\times M$, which satisfies the following properties;

$$\Delta_x \int G(x,y)f(y)dy = -f(x)$$

and

$$\int G(x,y)\Delta_y f(y)dy = -f(x),$$

for all smooth functions f with compact support on M. These two conditions are equivalent to that $G(x,y)$ satisfies the equation

$$\Delta_x G(x,y) = -\delta_x(y), \quad \text{for all } x\in M.$$

in distribution sense.

If M is R^n, then the explicit form of $G(x,y)$ is known. But in general, we can not expect explicit form of Green's function. Also the positivity of $G(x,y)$ is not guaranteed in general.

In 1955 Malgrange [2] showed that the Laplace operator admits a symmetric Green's function. But his argument was abstract and non-constructive. In case M admits a positive non-constant harmonic function, Yau and Schoen [3] proved that M admits a positive symmetric Green's function. In particular a complete manifold with lower bounded Ricci curvature admits a positive Green's function. Recently Li and Tam [1] constructed a Green's function, called a minimal Green's function.

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The purpose of this paper is to give a simplified version of Li and Tam's proof on the exitence of Green's function on a complete Riemannian manifold. That is, we are going to give another proof of the following theorem.

THEOREM. *Let M be a complete non-compact Riemannian manifold. Then there exists a Green's function on M.*

2. Proof of the exitence of Green's function

Let M be an n-dimensional complete noncompact Riemannian manifold without boundary. Consider a fixed point $p\in M$ and a monotone sequence of compact subdomains, Ω_i, which exhaust M. That is to say,

$$\Omega_j \supseteq \Omega_i, \quad \text{if } i<j,$$

and $\bigcup \Omega_i = M$.

For each i, we let $G_i(x,y)$ to be the symmetric Green's kernel on Ω_i which satisfies the Dirichlet boundary condition. It is known that $G_i(x,y)$ must behaves like

$$G_i \sim C(n) r(x,y)^{2-n}, \quad \text{as } y\to x, \quad \text{when } n>2,$$

and $G_i(x,y) \sim C(2) \log r(x,y)$, as $x\to y$, when $n=2$.

The costants $C(n)$ only depend on n, the dimension of M, and the function $r(x,y)$ denotes geodesic distance between x and y. The following lemma in Li and Tam [1] is essential in the proof

LEMMA. *Let p be a fixed point in M. The sequence of Green's functions $G_i(p,y)$ must have uniformly bounded oscillations in any compact subset K of $M-p$, for sufficiently large i's such that $\Omega_i \supset K$.*

Now we give our proof of the exitence of Green's function on a complete non-compact Riemannian manifold without boundary.

Proof. Let us define

$$l_i(r) = \inf\{G_i(p,y) \mid y\in \partial B_p(r)\}$$

and

$$S_i(r) = \sup\{G_i(p,y) \mid y\in \partial B_p(r)\}.$$

Let us denote $l_i(1)$ by a_i, By the lemma, for any given $R>1$, there exists a constant ω such that if i is sufficiently large then

$$S_i\left(\frac{1}{R}\right)\leq\omega+a_i \text{ and } a_i-\omega\leq l_i(R).$$

Applying the maximum principle, we have

$$-\omega\leq f_i(p,y)\leq\omega, \text{ on } B_p(R)-B_p\left(\frac{1}{R}\right)$$

where $f_i(p,y)=G_i(p,y)-a_i$.

Hence the $f_i(p,y)$'s are uniformly bounded on compact subsets of $M-p$, and there is a subsequence of the f_i's which converges uniformly on compact subsets of $M-p$.

Let x be a point in M, which is different from p. We are going to show that $f_i(x,y)$ as a function of y also converges on compact subsets of $M-p$.

As our previous argument, any subsequence of the $G_i(x,y)$'s contains a subsequence denoted by $G_j(x,y)$ and a set of non-negative numbers b_j's such that the sequence

$$G_j(x,y)-b_j$$

converges, Now, we show that we could take b_j's as a_j's, which implies that $f_i(x,y)$ as a function of y converges on compact subsets of $M-p$.

Since $f_i(p,y)$ converges to a function $G(p,y)$ implies that

$$\begin{aligned} G(p,x)=\lim_{i\to\infty} f_i(p,x)&=\lim_{j\to\infty}\{G_j(p,x)-a_j\} \\ &=\lim_{j\to\infty}\{G_j(p,x)-b_j\}+\lim_{j\to\infty}\{b_j-a_j\}. \end{aligned}$$

Hence the last limit must also converge to a constant c. Now clearly the subsequence

$$G_j(x,y)-a_j=\{G_j(x,y)-b_j\}+\{b_j-a_j\}$$

must also converge to some function $J(x,y)$. To show that the original sequence

$$f_i(x,y)=G_i(x,y)-a_i$$

converges, it suffices to prove that if there is another converging subsequence

$$G_k(x, y) - a_k$$

then it must converge to $J(x,y)$, Let us denote the limit by

$$\lim_{k\to\infty} \{G_k(x, y) - a_k\} = K(x, y).$$

The difference of the two functions $G(p, y)$ and $K(x, y)$ as functions of y must be bounded on $M - B_p(R)$ if $x \in B_p(R)$. In fact, let us consider the formula

$$G(p, y) - K(x, y) = \lim_{k\to\infty} \{G_k(p, y) - a_k\} - \lim_{k\to\infty} \{G_k(x, y) - a_k\}$$
$$= \lim_{k\to\infty} \{G_k(p, y) - G_k(x, y)\}$$

Due to the fact that G_k satisfies Dirichlet boundary condition, after applying the maximum principle, we have

$$\sup\{|G(p, y) - K(x, y)|| \ y \in M - B_p(R)\}$$
$$\leq \sup\{|G(p, y) - K(x, y)|| \ y \in \partial B_p(R)\},$$

which is bounded by the compactness of $\partial B_p(R)$ and the assumption that $x \in B_p(R)$. By the same argument, the difference of $G(p, y)$ and $J(x, y)$ is also a bounded function on $M - B_p(R)$. Hence the function

$$J(x, y) - K(x, y)$$

must also be bounded on $M - B_p(R)$. On the other hand, in view of the previous argument, we may assume that

$$a_k = \inf\{G_k(x, y) \mid y \in \partial B_x(1)\} - c,$$

where c is a constant. Therefore

$$H_{2R}(x, y) = c \leq G_k(x, y) - a_k + \theta \leq H_{2R}(x, y) + c + 2\theta,$$

with H_{2R} being the Dirichlet Green's function on $B_x(2R)$ and θ is the bound for the oscillation of the G_k's restricted on $B_x(2R) - B_x(1)$. Passing to the limit we conclude that $K(x, y)$ must behave like Green's function. Hence their difference $J(x, y) - K(x, y)$ is a bounded harmonic function on $B_p(R)$, therefore also a bounded harmonic function on M. The fact that there is no non-constant harmonic function on M now implies that $J(x, y) - K(x, y)$ is identically constant on M. Evaluating at $y = p$,

$$J(x, y)=\lim_{j\to\infty}\{G_j(x, y)-a_j\}=G(p, x)=\lim_{k\to\infty}\{G(x, p)-a_k\}=K(x, p),$$

which implies that $J(x, y)=K(x, y)$ for all $y\in M$.

This completes our proof.

References

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