Comm. Korean Math. Soc. 3(1988), No. 2, pp. 213~224

SEQUENTIAL YEH-FEYNMAN INTEGRALS OF CERTAIN CLASSES OF FUNCTIONALS

KUN SOO CHANG, JEONG GYOO KIM, IL YOO AND KI SEONG CHOI

1. Introduction

Let $C_2 \equiv C_2(Q)$ be the Yeh-Wiener space (or two parameter Wiener space) on $Q=[a,b]\times[c,d]$, that is, the space of continuous functions $x(s,t)$ on Q such that $x(s,c)=x(a,t)=0$. Let m_y be the Yeh-Wiener measure on $C_2(Q)$.

A subset E of $C_2(Q)$ is said to be scale-invariant measurable provided ρE is Yeh-Wiener measurable for every $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_y(\rho N)=0$ for every $\rho>0$ ([6]). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A function F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho(\cdot))$ is Yeh-Wiener measurable for every $\rho>0$. Two functionals F and G on $C_2(Q)$ are said to be equal s-a.e. ($F\approx G$) if for each $\rho>0$, the equation $F(\rho x)=G(\rho x)$ holds for a.e. x in $C_2(Q)$.

Let F be a functional such that the Yeh-Wiener integral

$$J(\lambda)=\int_{C_2} F(\lambda^{-1/2}x)dx$$

exists for all real $\lambda>0$. If there exists a function $J^*(\lambda)$ analytic in the half-plane $\operatorname{Re}\lambda>0$ such that $J^*(\lambda)=J(\lambda)$ for all real $\lambda>0$, then we define $J^*(\lambda)$ to be the analytic Yeh-Wiener integral of F over $C_2(Q)$ with parameter λ, and for $\operatorname{Re}\lambda>0$, we write

Received October 12, 1988.
Research supported in part by a grant from the Korea Science and Engineering Foundation and the Ministry of Education.

$$\int_{C_2}^{anyw_\lambda} F(x)\,dx = J^*(\lambda).$$

Let q be a non-zero real parameter and let F be a functional whose analytic Yeh-Wiener integral exists for $\operatorname{Re}\lambda>0$. Then if the following limit exists, we call it the analytic Yeh-Feynman integral of F over $C_2(Q)$ with parameter q, and we write

$$\int_{C_2}^{anyf_q} F(x)\,dx = \lim_{\substack{\lambda\to -iq \\ \operatorname{Re}\lambda>0}} \int_{C_2}^{anyw_\lambda} F(x)\,dx.$$

Let $Q=[a,b]\times[c,d]$ and let $a=s_0<s_1<\cdots<s_l=b$ and $c=t_0<t_1<\cdots<t_m=d$ determine a partition σ over Q. Let $f(s,t)$ be a real valued function defined on Q. A function $f(s,t)$ is said to be of bounded variation on $Q(f\in BV(Q))$ provided the following three conditions hold;

(i) there exists a constant K such that for any partition σ

$$(1.1)\quad \sum_{j=1}^{l}\sum_{k=1}^{m}|f(s_j,t_k)-f(s_{j-1},t_k)-f(s_j,t_{k-1})+f(s_{j-1},t_{k-1})|\leq K,$$

(ii) $f(s,d)$ is a function of bounded variation in s,

(iii) $f(b,t)$ is a function of bounded variation in t.

And the total variation of f over Q, $\operatorname{Var}(f,Q)$, is defined to be the supremum of the sums in (1.1) over all σ.

Let $f(s,t)$ be a real valued function on Q and let $R=[a',b']\times[c',d']$ be a subrectangle of Q and $\Delta_R(f)=f(b',d')-f(a',d')-f(b',c')+f(a',c')$. A function $f(s,t)$ is absolutely continuous on $Q(f\in AC(Q))$ if the following two conditions are satisfied;

(i) given $\varepsilon>0$, there exists $\delta>0$ suth that $\sum_{R\in S}|\Delta_R(f)|<\varepsilon$

whenever S is the finite collection of pairwise non-overlapping subrectangles of Q with $\sum_{R\in S}m(R)<\delta$, where m denotes Lebesgue measure on R^2,

(ii) the functions $f(\cdot,d)$ and $f(b,\cdot)$ are absolutely continuous functions of a single variable on $[a,b]$ and $[c,d]$, respectively.

In recent papers [5, 20], we treat some Banach algebras S, $\hat{S}$, and S^* of functionals on Yeh-Wiener space which are a kind of stochastic Fourier transform of complex Borel measures on $L_2(Q)$. Now we briefly review them.

Let $D_2 \equiv D_2(Q)$ be the class of elements $x \in C_2(Q)$ such that $x \in AC(Q)$ and $\partial^2 x(s,t)/\partial s \partial t \in L_2(Q)$. Let $\mathscr{M} \equiv \mathscr{M}(L_2(Q))$ be the class of complex measures of finite variation defined on $\mathscr{B}(L_2)$, the Borel measurable subsets of $L_2(Q)$. If $\mu \in \mathscr{M}$, we set $\|\mu\| = \text{var}\, \mu$ over L_2. (In this paper, L_2 always means real L_2.)

The functional F defined on a subset of $C_2(Q)$ that contains $D_2(Q)$ is said to be an element of $\hat{S} \equiv \hat{S}(L_2)$ if there exists a measure $\mu \in M$ such that for $x \in D_2(Q)$,

$$(1.2) \qquad F(x) = \int_{L_2} \exp\left\{ i \int_Q v(s,t) \frac{\partial^2 x(s,t)}{\partial s \partial t} ds dt \right\} d\mu(v).$$

The definition of the space of functionals S involves the Paley-Wiener-Zygmund (P.W.Z.) integral. Next we give the definition of the P.W.Z. integral, a simple type of stochastic integral, for functions of two variables.

Let $\{\phi_n\}$ be a complete orthonormal (C.O.N.) set of real valued functions of bounded variation on Q. Let $v \in L_2(Q)$ and

$$v_n(s,t) = \sum_{j=1}^{n} \phi_j(s,t) \int_Q v(p,q) \phi_j(p,q) dp dq.$$

Then the P.W.Z. integral with two parameters is defined by

$$\begin{aligned} \int_Q v(s,t) \tilde{d}x(s,t) &\equiv {}^{(\phi_n)}\int_Q v(s,t) \tilde{d}x(s,t) \\ &= \lim_{n \to \infty} \int_Q v_n(s,t) dx(s,t) \end{aligned}$$

The Riemann-Stieltjes integral $\int_Q v(s,t) dx(s,t)$ is then defined in the usual way [10]. A paper of Yeh [19] has a nice discussion of the n-dimensional Riemann-Stieltjes integral and some of its properties. Actually Yeh doesn't include conditions (ii) and (iii) as part of definition of bounded variation. Of course all of the results he obtains concerning the Riemann-Stieltjes integral are true in our more restrictive setting.

Let $S \equiv S(L_2)$ be the space of functionals F expressible in the form

$$(1.3) \qquad F(x) = \int_{L_2} \exp\{ i \int_Q v(s,t) \tilde{d}x(s,t) \} d\mu(v)$$

for s-a.e. x in C_2, where $\mu \in M$.

If $F(x)=G(x)$ for s-a.e. x in $C_2(Q)$ and for every x in $D_2(Q)$, we shall write $F\cong G$.

From Theorem 4 of [16], we have that if $v\in L_2(Q)$ and $x\in D_2(Q)$, then

$$(1.4)\qquad \int_Q v(s,t)\tilde{d}x(s,t)=\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\partial t}dsdt.$$

Thus if $v\in L_2(Q)$ and $\{\phi_n\}$, $\{\psi_n\}$ are two C.O.N. sequences of $BV(Q)$, then for $x\in D_2(Q)$,

$$\int_{L_2}\exp\{i\overset{\{\phi_n\}}{\int_Q} v(s,t)\tilde{d}x(s,t)\}d\mu(v)$$
$$\cong\int_{L_2}\exp\{i\overset{\{\psi_n\}}{\int_Q} v(s,t)\tilde{d}x(s,t)\}d\mu(v).$$

We now introduce the class of functionals S^*. Let $S^*\equiv S^*(L_2)$ be the space of functionals F expressible in the form

$$(1.5)\qquad F(x)=\int_{L_2}\exp\{i\int_Q v(s,t)\tilde{d}x(s,t)\}p\mu(v)$$

for s-a.e. $x\in C_2(Q)$ and for every $x\in D_2(Q)$, where $\mu\in M$.

2. Preliminaries and some results

In this section, we extend the concept of the sequential Feynman integral to that of the sequential Yeh-Feynman integral and prove the L_2 convergence of averaged functions.

NOTATION. Let a subdivision σ of Q be given;

$$\sigma: a=s_0<s_1<\cdots<s_l=b,\quad c=t_0<t_1<\cdots<t_m=d.$$

Let $x_\sigma\equiv x_\sigma((s,t),A)$ be a quadratic function in $C_2(Q)$ based on a subdivision σ and the matrix of real numbers $A=\{a_{j,k}\}$, and defined by

$$(2.1)\quad x_\sigma((s,t),A)=\frac{a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1}}{(s_j-s_{j-1})(t_k-t_{k-1})}(s-s_{j-1})(t-t_{k-1})$$
$$+\frac{a_{j,k-1}-a_{j-1,k-1}}{s_k-s_{k-1}}(s-s_{j-1})+\frac{a_{j-1,k}-a_{j-1,k-1}}{t_k-t_{k-1}}(t-t_{k-1})$$
$$+a_{j-1,k-1}$$

where $s_{j-1}\leq s\leq s_j$, $t_{k-1}\leq t\leq t_k$, $a_{j,0}=a_{0,k}=0$ for $j=1, \cdots, l$, $k=1, \cdots, m$.

As $A=\{a_{j,k}\}$ ranges over all lm-dimensional real space, the quadratic functions $x_\sigma((\cdot, \cdot), A)$ range over all quadratic approximations to the functions in $C_2(Q)$ based on σ. Specifically if x is a particular element of $C_2(Q)$ and we set $a_{j,k}=x(s_j, t_k)$, then the function $x_\sigma((\cdot, \cdot), A)$ is the quadratic approximation of x based on the subdivision σ.

DEFINITION 2. 1. Let q be a given non-zero real number and let $F(x)$ be a functional defined on a subset of $C_2(Q)$ containing all the quadratic elements of $C_2(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions such that norm $\|\sigma_n\|\to 0$, and let $\{\lambda_n\}$ be a sequence of complex numbers with $\operatorname{Re}\lambda_n>0$ such that $\lambda_n\to -iq$. Then if the integral in the right of (2. 2) exists for all n and if the following limit exists and is independent of the choice of the sequences of $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the sequential Yeh-Feynman integral with parameter q exists and is given by

$$(2.2)\qquad \int^{syf_q} F(x)dx$$
$$=\lim_{n\to\infty} \gamma_{\sigma_n,\lambda_n}\int_{R^{(lm)_n}} \exp\left\{-\frac{\lambda_n}{2}\int_Q\left[\frac{\partial^2 x_{\sigma_n}((s,t),A)}{\partial s\partial t}\right]^2 dsdt\right\}$$
$$F(x_{\sigma_n}((\cdot,\cdot),A))dA,$$

where

$$(2.3)\qquad \gamma_{\sigma,\lambda}=\left(\frac{\lambda}{2\pi}\right)^{lm/2}\left[\prod_{j=1}^{l}\prod_{k=1}^{m}(s_j-s_{j-1})(t_k-t_{k-1})\right]^{-1/2}$$

and $A=(A_1, \cdots, A_m)$, $A_k=(a_{1,k}, \cdots, a_{l,k})$ for $k=1, \cdots, m$.

We note that l, m depend on σ and lm is the number of subrectangles in σ. We empasize that the Lebesgue integral on the right of (2. 2) exists for all n.

Let

$$(2.4)\qquad W_\lambda(\sigma, A)=\gamma_{\sigma,\lambda}\exp\left\{-\frac{\lambda}{2}\int_Q\left[\frac{\partial^2 x_\sigma(s,t)}{\partial s\partial t}\right]^2 dsdt\right.$$
$$=\left(\frac{\lambda}{2\pi}\right)^{lm/2}\left[\prod_{j=1}^{l}\prod_{k=1}^{m}(s_j-s_{j-1})(t_k-t_{k-1})\right]^{-1/2}$$

$$\exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{[a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1}]^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}.$$

(By the notation $\lambda^{lm/2}$ we mean $(\sqrt{\lambda})^{lm}$ where $\operatorname{Re}\sqrt{\lambda}>0$). Thus in terms of W, the sequential Yeh-Feynman integral defined in (2.2) can be written

$$(2.5)\quad \int^{syf_q} F(x)dx=\lim_{n\to\infty}\int_{R^{(lm)_n}} W_{\lambda_n}(\sigma_n, A)F(x_{\sigma_n}((\cdot,\cdot), A))dA.$$

REMARK 2.1. Since $\{\sigma_n\}$ and $\{\lambda_n\}$ were chosen arbitrarily and independently in the definition, the single limit may also be expressed as a double limit, thus,

$$(2.6)\quad \int^{syf_q} F(x)dx=\lim_{n,k\to\infty} I_{n,k}$$

where

$$(2.7)\quad I_{n,k}=\int_{R^{(lm)_k}} W_{\lambda_n}(\sigma_k, A)F(x_{\sigma_k}((\cdot,\cdot), A))dA.$$

NOTATION. Let $v\in L_2(Q)$ and let σ be any subdivision such that

$$\sigma: a=s_0<s_1<\cdots<s_l=b,\quad c=t_0<t_1<\cdots<t_m=d.$$

We define the averaged function $v_\sigma(s,t)$ for v on σ by

$$(2.8)\quad v_\sigma(s,t)=\begin{cases}\dfrac{1}{(s_j-s_{j-1})(t_k-t_{k-1})}\displaystyle\int_{s_{j-1}}^{s_j}\int_{t_{k-1}}^{t_k} v(p,q)dpdq, \\ \quad \text{when } (s,t)\in[s_{j-1},s_j)\times[t_{k-1},t_k) \textit{ for} \\ \quad j=1,\cdots,l,\ k=1,\cdots,m. \\ 0, \text{ when } s=b \text{ or } t=d.\end{cases}$$

Where there is a sequence for subdivisions $\sigma_1, \sigma_2, \cdots$, then σ, l, m, s_j, and t_k will be replaced by σ_n, l_n, m_n, $s_{n,j}$, $t_{n,k}$.

The following proposition is a well known result. We will state it without proof [9, 17].

PROPOSITION 2.1. *Let* $Log^+(Q)$ *be the class of all functions* f *on* Q *such that* $|f|\log^+|f|$ *is integrable, where* $\log^+|f|=\log(|f|\vee 1)$. *Then* $L_2(Q)\subset\log^+(Q)$, *and hence for any* $v\in L_2(Q)$,

$$\lim_{h,k\to 0}\frac{1}{hk}\int_s^{s+h}\int_t^{t+k}v(p,q)\,dp\,dq=v(s,t)$$

for almost everywhere $(s,t)\in Q$.

PROPOSITION 2.2. *Let* $\{\sigma_n\}$ *be a sequence of subdivisions of* Q *such that the norm* $\|\sigma_n\|\to 0$, *and let* $v(s,t)$ *be integrable on* Q *such that the function* $|v|\log^+|v|$ *is integrable. Then for almost everywhere* $(s,t)\in Q$, *the sequence of averaged functions converges to the function;*

$$\lim_{n\to\infty}v_{\sigma_n}(s,t)=v(s,t). \tag{2.9}$$

Proof. By Proposition 2.1,

$$\lim_{h,k\to 0}\frac{1}{hk}\int_s^{s+h}\int_t^{t+k}v(p,q)\,dp\,dq=v(s,t) \tag{2.10}$$

for a.e. $(s,t)\in Q$. Let $(s^*,t^*)\in[a,b)\times[c,d)$ be a value of (s,t) where (2.10) holds, and hence

$$\lim_{h,k\to 0}\frac{1}{hk}\int_{s^*}^{s^*+h}\int_{t^*}^{t^*+k}v(p,q)\,dp\,dq=v(s^*,t^*).$$

Let $\varepsilon>0$ be given. Choose $\delta>0$ such that, when $0<|h|<\delta$ and $0<|k|<\delta$,

$$\left|\int_{s^*}^{s^*+h}\int_{t^*}^{t^*+k}v(p,q)\,dp\,dq-hk\,v(s^*,t^*)\right|\leq\varepsilon|hk|. \tag{2.11}$$

Choose N such that when $n>N$, $\|\sigma_n\|<\delta$. For j_n and k_n such that $s_{n,j_n-1}\leq s^*<s_{n,j_n}$, $t_{n,k_n-1}\leq t^*<t_{n,k_n}$, let us first put $h=s_{n,j_n}-s^*$ and $k=t_{n,k_n}-t^*$ in (2.11), Thus

$$\left|\int_{s^*}^{s_{n,j_n}}\int_{t^*}^{t_{n,k_n}}v(p,q)\,dp\,dq-(s_{n,j_n}-s^*)(t_{n,k_n}-t^*)v(s^*,t^*)\right|$$
$$\leq\varepsilon|(s_{n,j_n}-s^*)(t_{n,k_n}-t^*)|. \tag{2.11a}$$

Next put $h=s_{n,j_n}-s^*, k=t_{n,k_n-1}-t^*$; $h=s_{n,j_n-1}-s^*$, $k=t_{n,k_n}-t^*$; $h=s_{n,j_n-1}-s^*$, $k=t_{n,k_n-1}-t^*$, respectively. Thus

$$\left|\int_{s^*}^{s_{n,j_n}}\int_{t_{n,k_n-1}}^{t^*}v(p,q)\,dp\,dq-(s_{n,j_n}-s^*)(t^*-t_{n,k_n-1})v(s^*,t^*)\right|$$
$$\leq\varepsilon|(s_{n,j_n}-s^*)(t_{n,k_n-1}-t^*)|, \tag{2.11b}$$

$$(2.11c)\quad \left|\int_{s_{n,j_n-1}}^{s^*}\int_{t^*}^{t_{n,k_n}} v(p,q)\,dp\,dq-(s^*-s_{n,j_n-1})(t_{n,k_n}-t^*)v(s^*,t^*)\right| \leq \varepsilon|(s_{n,j_n-1}-s^*)(t_{n,k_n}-t^*)|,$$

$$(2.11d)\quad \left|\int_{s_{n,j_n-1}}^{s_*}\int_{t_{n,k_n-1}}^{t^*} v(p,q)\,dp\,dq-(s^*-s_{n,j_n-1})(t^*-t_{n,k_n-1})v(s^*,t^*)\right| \leq \varepsilon|(s_{n,j_n-1}-s^*)(t_{n,k_n-1}-t^*)|.$$

Thus we have

$$\left|\frac{1}{(s_{n,j_n}-s_{n,j_n-1})(t_{n,k_n}-t_{n,k_n-1})}\int_{s_{n,j_n-1}}^{s_{n,j_n}}\int_{t_{n,k_n-1}}^{t_{n,k_n}} v(p,q)\,dp\,dq-v(s^*,t^*)\right| \leq \varepsilon.$$

Therefore we have established

$$\lim_{n\to\infty} v_{\sigma_n}(s,t)=v(s^*,t^*).$$

PROPOSITION 2.3. *Let $\{\sigma_n\}$ be a sequence of subdivisions such that the norm $\|\sigma_n\|\to 0$, and let $v\in L_2(Q)$. Then*

$$\lim_{n\to\infty}\int_Q [v_{\sigma_n}(s,t)]^2 ds\,dt=\int_Q [v(s,t)]^2 ds\,dt.$$

Proof. By Proposition 2.1 and 2.2, $\lim_{n\to\infty} v_{\sigma_n}(s,t)=v(s,t)$ for a.e. $(s,t)\in Q$, and by Fatou's lemma,

$$\lim_{n\to\infty}\inf\int_Q [v_{\sigma_n}(s,t)]^2 ds\,dt\geqq\int_Q [v(s,t)]^2 ds\,dt.$$

Now

$$v_{\sigma_n}(s,t)=\frac{1}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(p,q)\,dp\,dq$$

when $(s,t)\in[s_{n,j-1},s_{n,j})\times[t_{n,k-1},t_{n,k})$, and by Schwarz ineqality,

$$(2.12)\quad \left[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(p,q)\,dp\,dq\right]^2 \leqq\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} [v(p,q)]^2 dp\,dq\,(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1}).$$

Therefore we have

$$(2.13)\qquad \int_Q [v_{\sigma_n}(s,t)]^2 dsdt$$

$$=\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\left[\frac{\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(p,q)\,dsdq}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right]^2 (s_{n,j}-s_{n,j-1})$$

$$(t_{n,k}-t_{n,k-1})\leq\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}}[v(p,q)]^2 dpdq$$

$$=\int_Q [v(s,t)]^2 dsdt.$$

Thus

$$\limsup_{n\to\infty}\int_Q [v_{\sigma_n}(s,t)]^2 dsdt\leqq\int_Q [v(s,t)]^2 dsdt.$$

We have proved that the lim sup is less than or equal to the lim inf and so the proposition is proved.

3. Existence theorem of the sequential Yeh-Feynman integral on $\hat{S}$ and S^*

In this section, we prove the existence of the sequential Yeh-Feynman integral for every element of each of our Banach algebras $\hat{S}, S^*$. And also we present a theorem for interchanging summation and sequential Yeh-Feynman integration.

THEOREM 3.1. *If $F\in S^*$ and q is a non-zero real number, then F is sequentially Yeh-Feynman integrable and its sequential Yeh-Feynman integral is equal to its analytic Yeh-Feynman integral.*

Proof. Since $F\in S^*$, there exists a measure $\mu\in M$ such that

$$(3.1)\qquad F(x)\simeq\int_{L_2}\exp\left\{i\int_Q v(s,t)\tilde{d}x(s,t)\right\}d\mu(v).$$

In particular, this equality holds for all quadradic functions x_{σ_n}. Let $\{\sigma_n\}$ and $\{\lambda_n\}$ be such that the norm $\|\sigma_n\|\to 0$, and Re $\lambda_n>0$ and $\lambda_n\to -iq$. Then

$$J_n\equiv\int_{R^{(lm)_n}} W_{\lambda_n}(\sigma_n, A)F(x_{\sigma_n}((\cdot,\cdot),A))dA$$

$$=\int_{R^{(lm)_n}} W_{\lambda_n}(\sigma_n, A)\int_{L_2}\exp\{i\int_Q v(s,t)\tilde{d}x_{\sigma_n}(s,t)\}\,d\mu(v)dA.$$

By Fubini theorem and the properties of the P.W.Z. integral, we have

$$J_n=\int_{L_2}\int_{R^{(lm)_n}} W_{\lambda_n}(\sigma_n, A)\exp\{i\int_Q v(s,t)\frac{\partial^2 x_{\sigma_n}((s,t),A)}{\partial s\partial t}dsdt\}dAd\mu(v)$$

$$=\gamma_{\sigma_n,\lambda_n}\int_{L_2}\int_{R^{(lm)_n}}\exp\Big\{-\frac{\lambda_n}{2}\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{[a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1}]^2}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\Big\}$$

$$\exp\Big\{i\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(s,t)$$

$$\frac{[a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1}]}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}dsdt\Big\}dAd\mu(v)$$

$$=\gamma_{\sigma_n,\lambda_n}\int_{L_2}\int_{R^{(lm)_n}}\exp\Big\{-\frac{\lambda_n}{2}\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{b_{j,k}^2}{(s_{n,j-1}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\Big\}$$

$$\exp\Big\{i\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(s,t)\frac{b_{j,k}}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}dsdt\Big\}$$

$$dBd\mu(v)$$

where $B=\{b_{j,k}\}$ which $b_{j,k}=a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1}$. Thus

$$J_n=\gamma_{\sigma_n,\lambda_n}\int_{L_2}\int_{R^{(lm)_n}}\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\exp\Big\{-\frac{\lambda_n b_{j,k}^2}{2(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}$$

$$+i\Big[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(s,t)dsdt\Big]\frac{b_{j,k}}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\Big\}dBd\mu(v)$$

$$=\gamma_{\sigma_n,\lambda_n}\int_{L_2}\prod_{j=1}^{l_n}\prod_{k=1}^{m_n}\Big[\frac{2\pi(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}{\lambda_n}\Big]^{1/2}$$

$$\exp\Big\{-\frac{\Big[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(s,t)dsdt\Big]^2}{2\lambda_n(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\Big\}d\mu(v)$$

$$=\int_{L_2}\exp\Big\{-\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{\Big[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v(s,t)dsdt\Big]^2}{2\lambda_n(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\Big\}d\mu(v).$$

$$=\int_{L_2}\exp\Big\{-\frac{1}{2\lambda_n}\int_Q[v_{\sigma_n}(s,t)]^2dsdt\Big\}d\mu(v).$$

Thus we have

$$J_n=\int_{L_2}\exp\Big\{-\frac{1}{2\lambda_n}\int_Q[v_{\sigma_n}(s,t)]^2dsdt\Big\}d\mu(v)$$

$$\to\int_{L_2}\exp\Big\{\frac{1}{2qi}\int_Q[v(s,t)]^2dsdt\Big\}d\mu(v)$$

as $n\to\infty$, by Proposition 2.3 and the bounded convergence theorem. Therefore we have proved that

$$\lim_{n\to\infty} J_n = \int^{syf_q} F(x)dx = \int_{L_2} \exp\left\{\frac{1}{2qi}\int_Q [v(s,t)]^2 dsdt\right\} d\mu(v)$$

Since $F\in S^*\subset S$, we have by proposition 3.1 in [5] that the last member above is equal to the analytic Yeh-Feynman integral of F.

COROLLARY 1. *If $F\in S^*$, q is a non-real number, and F is given by* (1.5) *where $\mu\in M$, then*

$$(3.2)\quad \int^{syf_q} F(x)dx = \int_{L_2} \exp\left\{\frac{1}{2qi}\|v\|_2^2\right\} d\mu(v) = \int_{C_2}^{anyf_q} F(x)dx.$$

COROLLARY 2. *Ler $F\in\hat{S}$. Then F is sequentially Yeh-Feynman integrable and the first two members of* (3.2) *are equal.*

We also prove that the sequential Yeh-Feynman integration can be interchanged with infinite summation.

THEOREM 3.2. *Let $F_n\in S^*$ for $n=1,2,\cdots$, and let $\sum_{n=1}^{\infty}\|F_n\|<\infty$. Then $F\in S^*$, where $F(x)\simeq\sum_{n=1}^{\infty}F_n(x)$, and*

$$\int^{syf_q} F(x)dx = \sum_{n=1}^{\infty}\int^{syf_q} F_n(x)dx.$$

Proof. This follows from Proposition 3.2 in [5] and Theorem 3.1

References

1. S. Albeverio and R. Høegh-Krohn, *Mathematical theory of Feynman path integrals*, Lecture Notes in Math. **523**, Springer-Verlag, 1976.
2. E. Berkson and T.A. Gillespie, *Absolutely continuous functions of two variables and well-bounded operators*, J. London Math. Soc. **30**(1984), 305–321.
3. R.H. Cameron and D.A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*, Lecture Notes in Math. **798**, Springer-Verlag, 1980.
4. R.H. Cameron and D.A. Storvick, *A simple definition of the Feynman integral, with applications*, Mem. Amer. Math. Soc. **288**(1983).
5. K.S. Chang, J.M. Ahn and J.S. Chang, *Notes on the analytic Yeh-Feynman integrable functionals*, Rocky Mountain J. of Math. **18**(1988), 157–165.

6. K.S. Chang, *Scale-invariant measurability in Yeh-Wiener space*, J. of Korean Math. Soc., **19**(1982), 61-67.
7. K.S. Chang, G.W. Johnson and D.L. Skoug, *The Feynman integral of quadratic potentials depending on two time variables*, Pacific J. Math. **122**(1986), 11-33.
8. R.P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. **20**(1948), 367-387.
9. L.M. Graves, *The theory of functions of real variables*, McGraw-Hill, 1956.
10. E.W. Hobson, *The theory of functions of a real variables and the theory of Fourier series*, Vol. **1**, Dover, New York, 1957.
11. G.W. Johnson and D.L. Skoung, *Scale-invariant measurability in Wiener space*, Pacific J. Math. **83**(1979), 157-176.
12. G.W. Johnson and D.L. Skoug, *A stochastic integration by parts formula for two-parameter Wiener × two-parameter Wiener space*, SIAM J. Math. Anal. **18**(1987), 919-932.
13. J. Kuelbs, *Integration on spaces of continuous functions*, Thesis, University of Minnesota, 1965.
14. J. Kuelbs and V. Mandrekar, *Harmonic analysis on F-spaces with a Basis*, Trans. Amer. Math. Soc. **169**(1972), 113-152.
15. C. Park, *Generalized Riemann-Stieltjes integral over the K.Y.W. space of functions of two variables*, Thesis, University of Minnesota, 1968.
16. C. Park, *A generalized Paley-Wiener-Zygmund integral and its applications*, Proc. Amer. Math. Soc. **23**(1969), 388-400.
17. S. Saks, *Theory of the integral*, Hafner Publishing company, Now York, 1937.
18. J. Yeh, *Wiener measure in a space of functions of two variables*, Trans. Amer. Math. Soc. **95**(1960), 433-450.
19. J. Yeh, *Cameron-Martin translation theorems in the Wiener space of functions of two variables*, Trans. Amer. Math. Soc. **107**(1963), 409-420.
20. I. Yoo, *Sequential Yeh-Feynman integrals*, Thesis, Yonsei University, 1986.

Yonsei University
Seoul 120-749, Korea