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EXTENSIONS OF THE CUNTZ ALGEBRAS RELATIVE TO A SEMIFINITE DECOMPOSABLE FACTOR*

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1. Introduction

In 1909, H. Weyl showed that every self-adjoint operator on a separable infinite dimensional Hilbert space $\mathscr{H}$ is a diagonal plus compact. In 1935, von Neumann proved that two self-adjoint operators on $\mathscr{H}$ are unitarily equivalent up to compacts if and only if they have the same spectrum up to isolated eigenvalues of finite multiplicity. Then Berg and Sikonia extended this result to normal operators.

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$, $\mathscr{K}(\mathscr{H})$ the two-sided ideal of compact operators, $Q(\mathscr{H})$ the quotient algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, and π the canonical homomorphism of $\mathscr{L}(\mathscr{H})$ onto $Q(\mathscr{H})$. An operator N is called *essentially normal* if $\pi(N)$ is normal in $Q(\mathscr{H})$. Note that for normal operator N the spectrum of $\pi(N)$ in $Q(\mathscr{H})$ is the same as the spectrum of N in $\mathscr{L}(\mathscr{H})$ minus eigenvalues of finite multiplicity. Thus one would hope that two essentially normal operators N_1 and N_2 are unitarily equivalent up to compacts if and only if $\pi(N_1)$ and $\pi(N_2)$ have the same spectrum in $Q(\mathscr{H})$. But this is not so in general.

Brown, Douglas, and Fillmore(BDF for short) [3, 4] proved that Fredholm index data is a complete invariant for the classification of essentially normal operators. In doing so, BDF found a beautiful theory, the so-called BDF theory, which connects the operator theory on the one end and the seemingly unrelated algebraic topology on the other end. BDF theory has been generalized to non-commutative C^*-algebras by many authors.

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It is well-known that a semifinite decomposable von Neumann factor has very similar properties with $\mathscr{L}(\mathscr{H})$. In fact, some of BDF theory has been extended to this context [5, 8, 9, 12]. In this note we study the unitary equivalence classes of unital $*$-monomorphisms of the Cuntz algebra to the generalized Calkin algebra of a semifinite decomposable von Neumann factor.

2. Preliminaries

Let $\mathscr{M}$ be a semifinite factor acting on a separable Hilbert space $\mathscr{H}$. An operator P is called *projection* if it is a self-adjoint idempotent, i.e. $P=P^*=P^2$. An operator U is called *partial isometry* if both U^*U and UU^* are projections. For partial isometry U, U^*U and UU^* are called the *initial projection* and *final projection* of U, respectively. Two projections P, Q in $\mathscr{M}$ are *equivalent* if there exists a partial isometry U in $\mathscr{M}$ such that $P=U^*U$ and $Q=UU^*$. Then it is routine to check that this is indeed an equivalence relation on the set of all projections $\mathscr{P}(\mathscr{M})$ of $\mathscr{M}$. The equivalence of two projections P and Q will be denoted by $P\sim Q$. A projection P in $\mathscr{M}$ is *finite* if no proper subprojection of P is equivalent to P. A projection is *infinite* if it is not finite. Hence a projection P is infinite if and only if there exists a proper subprojection P' which is equivalent to P. There exists a nonnegative extended real-valued function on $\mathscr{P}(\mathscr{M})$ which resembles the usual dimension function of Hilbert spaces. More precisely, there is a function dim on $\mathscr{P}(\mathscr{M})$ with range $[0, \infty]$ such that

(i) $P\sim Q$ if and only if $\dim(P)=\dim(Q)$

(ii) if P and Q are orthogonal, then $\dim(P+Q)=\dim(P)+\dim(Q)$

(iii) P is finite if and only if $\dim(P)<\infty$

(iv) P is infinite if and only if $\dim(P)=\infty$.

We mention that such a dimension function on $\mathscr{P}(\mathscr{M})$ is unique up to constant multiples.

Let $\mathscr{K}(\mathscr{M})$ be the norm-closed two sided $*$-ideal of $\mathscr{M}$ generated by all finite projections of $\mathscr{M}$. This closed ideal $\mathscr{K}(\mathscr{M})$ resembles in many respects the usual compact ideal of the algebra $\mathscr{L}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$. This ideal $\mathscr{K}(\mathscr{M})$ is the only non trivial closed two sided ideal of $\mathscr{M}$. Let π denote the canonical homomorphism of $\mathscr{M}$ onto $\mathscr{M}/\mathscr{K}(\mathscr{M})$. We now briefly describe Fredholm operator relative to $\mathscr{M}$ and its associated relative index. These generalization to semifinite factor context was mainly done by Breuer. Details can be found in [1, 2].

An operator T in $\mathscr{M}$ is *Fredholm* relative to $\mathscr{M}$ if the projection on its null space is finite and if there exists a cofinite projection in $\mathscr{M}$ with range contained in $T(\mathscr{H})$. Then we have the following generalization of Atkinson's Theorem due to Breuer [1, 2].

THEOREM A. *An operator T in $\mathscr{M}$ is Fredholm relative to $\mathscr{M}$ if and only if $\pi(T)$ is invertible in $\mathscr{M}/\mathscr{K}(\mathscr{M})$.*

Let N_T denote the null projection of T in $\mathscr{M}$. For Fredholm operator T in $\mathscr{M}$, both $\dim(N_T)$ and $\dim(N_{T^*})$ are finite. Hence we can define

$$\mathrm{ind}_m(T)=\dim(N_T)-\dim(N_{T^*})$$

for Fredholm $T\in\mathscr{M}$.

Since dimension function on $\mathscr{M}$ is unique up to constant multiples, so is index function. For details of index map, see [1, 2, 10].

We now consider extensions of C^*-algebras by the generalized compact ideal $\mathscr{K}(\mathscr{M})$. Let $\mathscr{A}$ be a C^*-algebra. Consider extension of the form of short exact sequence

$$0\to\mathscr{K}(\mathscr{M})\to\mathscr{E}\to\mathscr{A}\to 0.$$

Such an extension of $\mathscr{A}$ by $\mathscr{K}(\mathscr{M})$ is equivalent to a $*$-homomorphism

$$\tau:\mathscr{A}\to\mathscr{M}/\mathscr{K}(\mathscr{M}).$$

DEFINITIONS. (i) An extension is a unital $*$-monomorphism $\tau:\mathscr{A}\to\mathscr{M}/\mathscr{K}(\mathscr{M})$.

(ii) An extension τ is *trivial* if τ can be factored through $\mathscr{M}$, that is, if there exists a unital $*$-homomorphism $\sigma:\mathscr{A}\to\mathscr{M}$ such that $\tau=\tau\circ\sigma$.

(iii) The sum of two extensions τ_1 and τ_2 is the extension defined as follows:

$$(\tau_1\oplus\tau_2)(x)=\begin{pmatrix}\tau_1(x) & 0\\ 0 & \tau_2(x)\end{pmatrix}\in\mathscr{M}_2(\mathscr{M}/\mathscr{K}(\mathscr{M}))\simeq\mathscr{M}/\mathscr{K}(\mathscr{M})$$

for all $x\in\mathscr{A}$.

(iv) Two extenions τ_1 and τ_2 are *unitarily equivalent* if there exists a unitary $U\in\mathscr{M}$ such that for all $x\in\mathscr{A}$

$$\tau_2(x)=\pi(U)^*\tau_1(x)\pi(U).$$

(v) Two extensions τ_1 and τ_2 are *equivalent* if theire exist trivial extensions ϕ_1 and ϕ_2 such that $\tau_1\oplus\phi_1$ and $\tau_2\oplus\phi_2$ are unitarily equivalent.

Let $\mathrm{Ext}_{\mathscr{M}}(\mathscr{A})$ denote the equivalence classes of all extensions of $\mathscr{A}$. Then obviously $\mathrm{Ext}_{\mathscr{M}}(\mathscr{A})$ forms a semi-group with equivalence class of trivial extensions as the zero element. Moreover, if $\mathscr{A}$ is a separable C^*-algebra, then by Choi-Effros's Lifting Theorem [6] $\mathrm{Ext}_{\mathscr{M}}(\mathscr{A})$ is a group.

THEOREM B. *Let $\mathscr{A}$ be a separable nuclear C^*-algebra. Then $Ext_{\mathscr{M}}(\mathscr{A})$ is an abelian group.*

We close this section with a couple of comments on the extension group relative to $\mathscr{M}$. First of all, trivial extensions of the classical compact ideal are all unitarily equivalent. But it is not clear whether trivial extensions of the generalized compact ideal $\mathscr{K}(\mathscr{M})$ are all unitarily equivalent. Elliott and Takemoto[8] showed that for AF algebras all trivial extensions are unitarily equivalent. The above definition of stable equivalence relation was due to Skandalis [12]. It is of some interest to determine the dependence of the extension group $\mathrm{Ext}_{\mathscr{M}}\mathscr{A}$ with respect to semifinite factor $\mathscr{M}$.

3. The Main Results

Cuntz [7] studied the C^*-algebra $\mathscr{O}_n$ generated by isometries S_i on $\mathscr{H}$ with $\sum_{i=1}^{n}S_iS_i^*=1$ for natural number n. If n is infinite, $\mathscr{O}_\infty$ is the C^*-algebra generated by infinite number of isometries with $\sum_{i=1}^{n}S_iS_i^*\leqq 1$ for all natural number n. Among other things, he showed that the C^*-algebra $\mathscr{O}_n$ is independent of the isometries $S_1, \cdots, S_n$.

Pimsner and Popa [11] computed the extension group $\mathrm{Ext}\,\mathscr{O}_n$. In this section with use of generalized Fredholm index and similar technique of liftings as in [11] we compute the extension group $\mathrm{Ext}_{\mathscr{M}}\mathscr{O}_n$ relative to a semifinite factor $\mathscr{M}$.

We begin with the following Lemma.

LEMMA 1. *Let u be an isometry in $\mathscr{M}/\mathscr{K}(\mathscr{M})$ and P a projection* *in* $\mathscr{M}$ *with* $\pi(P)=uu^*$. *Then there exists a partial isometry* U in $\mathscr{M}$ such that $\pi(U)=u$ and $UU^*\leqq P$.

Proof. Choose X in $\mathscr{M}$ with $\pi(X)=u$. Since $\pi(PX)=\pi(P)\pi(X)=uu^*u=u$, we may assume that the range of X is contained in $P\mathscr{H}$. Let $X=U(X^*X)^{1/2}$ be its polar decomposition. Then both U and $(X^*X)^{1/2}$ are in $\mathscr{M}$. Since $\pi((X^*X)^{1/2})=(\pi(X^*X))^{1/2}=(u^*u)^{1/2}=1$, we have $\pi(X)=\pi(U)$. Furthermore, since the range of U is contained in $P\mathscr{H}$, $UU^*\leqq P$. Thus U is a partial isometry with the desired properties.

DEFINITION. Let P be a projection in $\mathscr{M}$. Let U be a partial isometry in $\mathscr{M}$ with $UU^*\leqq P$, $\pi(UU^*)=\pi(P)$, and $\pi(U^*U)=1$. Then the relative index, $\mathrm{ind}(U,P)$, of U with respect to P is the number

$$\mathrm{ind}(U,P)=\dim(1-U^*U)-\dim(P-UU^*).$$

LEMMA 2. *Let* U *and* P *as above in Definition.*

(i) *If* $ind(U,P)=0$, *then there exists an isometry* V *in* $\mathscr{M}$ *with*

$$\pi(V)=\pi(U) \text{ and } VV^*=P.$$

(ii) *If* $ind(U,P)<0$, *then there exists an isometry* V *in* $\mathscr{M}$ *with*

$$ind(U,P)=-dim(P-VV^*)$$

(iii) *If* $ind(U,P)>0$, *then there exists a coisometry* V *with*

$$ind(U,P)=dim(1-V^*V).$$

Proof. (i) Since $\mathrm{ind}(U,P)=0$, $\dim(1-U^*U)=\dim(P-UU^*)$. Therefore two projections $1-U^*U$ and $P-UU^*$ are equivalent. Hence there is a partial isometry $\tilde{U}$ in $\mathscr{M}$ with $\tilde{U}^*\tilde{U}=1-U^*U$ and $\tilde{U}\tilde{U}^*=P-UU^*$. Thus $\tilde{U}$ is in $\mathscr{K}(\mathscr{M})$ and $\pi(U+\tilde{U})=\pi(U)$. Thus $U+\tilde{U}$ is an isometry with the desired property.

(ii) Since $\mathrm{ind}(U,P)<0$, $\dim(1-U^*U)<\dim(P-UU^*)$. Hence there exists a subprojection Q of P such that $1-U^*U\sim Q<P-UU^*$. Take a partial isometry $\tilde{U}$ with $\tilde{U}^*\tilde{U}=1-U^*U$ and $\tilde{U}\tilde{U}^*=Q$. Then $U+\tilde{U}$ is an isometry with the desired property.

(iii) It can be proved similarly as (ii).

LEMMA 3. *Let* $s_1,\cdots,s_n$ be isometries in $\mathscr{M}/\mathscr{K}(\mathscr{M})$ *with* $s_1s_1^*+\cdots$

$+s_ns_n^*=1$. *Let* $P_1, \cdots, P_n$ *be projections in* $\mathscr{M}$ *with* $P_1+P_2+\cdots+P_n=1$ *and* $\pi(P_i)=s_is_1^*$ *for* $i=1, 2, \cdots, n$. *Let* $S_1, \cdots, S_n$ *be partial isometries in* $\mathscr{M}$ *with* $\pi(S_i)=s_i$ *and* $S_iS_i^*\leqq P_i$ *for* $i=1, 2, \cdots, n$. *Then* $ind(S_1, P_1)+ind(S_2, P_2)+\cdots+ind(S_n, P_n)$ *is independent of the choices of* S_i *and* P_i.

Proof. Let U_i be isometries with $U_iU_i^*=P_i$. Then $\operatorname{ind}(S_i, P_i)=-\operatorname{ind}(U_iS_1^*)$, relative index considered as an operator on $P_i\mathscr{H}$. Therefore

$$\sum_{i=1}^{n}\operatorname{ind}(S_i, P_i)=-\sum_{i=1}^{n}\operatorname{ind}(U_iS_i^*)=\operatorname{ind}(\sum_{i=1}^{n}U_iS_i^*).$$

Let T_i, Q_i be isometries and projections with the stated condition, respectively. Then $\operatorname{ind}(T_i, Q_i)=-\operatorname{ind}(U_i, T_i^*)$, relative index considered as an operator from $Q_i\mathscr{H}$ to $P_i\mathscr{H}$, i.e., $\operatorname{ind}(U_iT_i^*)=\dim(Q_i-N_{U_iT_i^*})-\dim(P_i-N_{T_iU_i^*})$. Since

$$\pi(U_1S_1^*+\cdots+U_nS_n^*)=\pi(U_1)\pi(S_1^*)+\cdots+\pi(U_n)\pi(S_n^*)$$
$$=\pi(U_1)\pi(T_1^*)+\cdots+\pi(U_n)\pi(T_n^*)=\pi(U_1T_1^*+\cdots+U_nT_n^*),$$

we have $\operatorname{ind}(\sum_{i=1}^{n}U_iS_i^*)=\operatorname{ind}(\sum_{i=1}^{n}U_iT_i^*)$. Hence $\sum_{i=1}^{n}\operatorname{ind}(S_i, P_i)=-\operatorname{ind}(\sum_{i=1}^{n}U_iS_i^*)=-\operatorname{ind}(\sum_{i=1}^{n}U_iT_i^*)=\sum_{i=1}^{n}\operatorname{ind}(T_i, Q_i)$, which completes the proof.

Let T_i be isometries in $\mathscr{O}_n$ with $T_1T_1^*+\cdots+T_nT_n^*=1$. Let s_i be isometries in $\mathscr{M}/\mathscr{K}(\mathscr{M})$ with $s_1s_1^*+\cdots+s_ns_n^*=1$. Then by sending T_i to s_i we have a unital $*$-monomorphism from $\mathscr{O}_n$ to $\mathscr{M}/\mathscr{K}(\mathscr{M})$, and vice versa. Thus by examining liftings of s_i to $\mathscr{M}$ we can determine the extension group $\operatorname{Ext}_{\mathscr{M}}\mathscr{O}_n$.

THEOREM 1. *For each natural number n, the extension group of the Cuntz algebra* $\mathscr{O}_n$ *is isomorphic to the additive group* $\boldsymbol{R}$ *of all real numbers.*

Proof. Let $\tau: \mathscr{O}_n\to\mathscr{M}/\mathscr{K}(\mathscr{M})$ be a unital $*$-monomorphism. By repeating Lemma 1, we can find S_i's and P_i's in $\mathscr{M}$ such that $\pi(S_i)=\tau(T_i)$, $\pi(P_i)=(T_iT_i^*)$, and $S_iS_i^*\leqq P_i$. Then defiine

$$\theta(\tau)=\operatorname{ind}(S_1, P_1)+\cdots+\operatorname{ind}(S_n, P_n).$$

By Lemma 3, $\sum_{i=1}^{n}\operatorname{ind}(S_i, P_i)$ is independent of choices of S_i and P_i. Also, it is easy to see that if τ and τ' are unitarily equivalent then $\theta(\tau)$

$=\theta(\tau')$. Also note that if $S_iS_i^*=P_i$ and $S_i^*S_i=1$ then $\mathrm{ind}(S_i, P_i)=0$. Hence $\theta(\tau)=\theta(\tau+\tau_0)$, where τ_0 is a trivial extension. Thus θ is a well-defined homomorphism from $\mathrm{Ext}_{\mathscr{M}}\mathscr{O}_n$ to $\boldsymbol{R}$. Now suppose that $\theta(\tau)=0$. Then with help of Lemma 2 and Lemma 3 by adding and subtracting finite projections if necessary we may assume that $\mathrm{ind}(S_i, P_i)=0$ for all $i=1, 2, \cdots, n$. Then by Lemma 2(i), τ is a trivial extension. Hence $[\tau]=0$. To prove the surjectivity of θ, for any real number r choose a partial isometry S_1 and a projection P_1 such that $\mathrm{ind}(S_1, P_1)=r$. Then for $i=2, \cdots, n$ choose isometries S_i and projections P_i such that $S_iS_i^*=P_i$ and $P_1+P_2+\cdots+P_n=1$. Let τ be the extension determined by sending T_i to $\pi(S_i)$. Then by construction $\theta(\tau)=\mathrm{ind}(S_1, P_1)=r$. Thus θ is an isomorphism of $\mathrm{Ext}_{\mathscr{M}}\mathscr{O}_n$ onto $\boldsymbol{R}$.

Next we compute the extension group of $\mathscr{O}_\infty$ relative to semifinite factor $\mathscr{M}$.

LEMMA 4. *For $i=1, 2, \cdots$ let s_i be isometries in $\mathscr{M}/\mathscr{K}(\mathscr{M})$ with $\sum_{i=1}^{n} s_is_i^*\leqq 1$ for all n. Then there exist isometries S_i in $\mathscr{M}$ with $\pi(S_i)=s_i$ and $\sum_{i=1}^{n} S_iS_i^*\leqq n$ for all n.*

Proof. Choose isometry S_1 in $\mathscr{M}$ with $\pi(S_1)=s_1$ and $1-S_1S_1^*$ infinite projection. Suppose $S_1, \cdots, S_n$ have been chosen so that $\pi(S_i)=s_i$, $\sum_{i=1}^{n} S_iS_i^*<1$, and $1-\sum_{i=1}^{n} S_iS_i^*$ infinite. Then by Lemma 1 one can choose an isometry S_{n+1} in $\mathscr{M}$ with $\pi(S_{n+1})=s_{n+1}$, $S_{n+1}S_{n+1}^*\leqq 1-(S_1S_1^*+\cdots+S_nS_n^*)$, and $1-S_{n+1}S_{n+1}^*$ infinite. This completes the proof.

THEOREM 2. *The extension group of the Cuntz algebra $\mathscr{O}_\infty$ is trivial, that is, $\mathrm{Ext}_{\mathscr{M}}\mathscr{O}_\infty=\{0\}$.*

Proof. Let $\tau: \mathscr{O}_\infty\to\mathscr{M}/\mathscr{K}(\mathscr{M})$ be any extension. Then by Lemma 4, isometries $\tau(T_i)$ can be lifted to isometries S_i in $\mathscr{M}$. Let σ be the $*$-isomorphism determined by sending T_i to S_i. Then by construction $\tau=\pi\circ\sigma$. Hence τ is trivial. This completes the proof.

References

1. M. Breuer, *Fredholm theories on von Neumann algebras I*, Math. Ann. **178** (1968), 243-254.
2. M. Breuer, *Fredholm theories on von Neumann algebras II*, Math. Ann. **180** (1969), 313-325.
3. L.G. Brown, R.G. Douglas and P.A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proc. Conf. on Operator Theory, Springer-Verlag Lecture Notes **345**, 58-128, 1973.
4. _____, *Extensions of C^*-algebras and K-homology*, Ann. Math. (2) **105**(1977), 265-324.
5. S.J. Cho, *Extensions relative to a II_∞ factor*, Proc. Amer. Math. Soc. **74** (1979), 109-112.
6. M.D. Choi and E.G. Effros, *The completely positive lifting problem*, Ann. Math. **104**(1976), 585-609.
7. J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173-185.
8. G.A. Elliott and H. Takemoto, *On C^*-algebra extensions relative to a factor of type II_∞*, preprint.
9. P.A. Fillmore, *Extensions relative to semi-finite factors*, Symp. Math. **20** Inst. Naz di Alta Mat. (1976) 487-496.
10. C.L. Olsen, *Index theory in von Neumann Algebras*, Memoirs Amer. Math. Soc. **47**(1984), No. 294.
11. M. Pimsner and S. Popa, *The Ext-groups of some C^*-algebras considered by J. Cuntz*, Rev. Roum. Math. Pures Appl. **23**(1978), 1069-1076.
12. G. Skandalis, *On the group of extensions relative to a semifinite factor*, Queen's Mathematical Preprint No. 1983-1.

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