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ON NORMALIZED TOPOLOGICAL DIVISORS OF ZERO ON BANACH*-ALGEBRA

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1. Introduction

In [4], M. Fujii and S. Lin introduced the normalized topological divisors of zero and studied some characterizations of normal approximate spectra. The object of this paper is to study the inclusion relations among the set of normalized topological divisors of zero, spectrum and the left and right topological divisors of zero for a Banach*-algebra. Thus we obtain some generalizations of these sets for the B^*-algebra. In the last section, we obtain a generalization of Gelfand-Majur Theorem and give elementary proofs of well known results in the Banach algebra Theory.

Since every B^*-algebra is isometrically *-isomorphic to a C^*-algebra of bounded operators on some Hilbert spaces, so the above results for spectra apply to the spectrums of the algebra of bounded operators on some Hilbert space.

Throughout this paper, all algebras and vector spaces will be over the complex field C. Algebras are assumed to have an unity element, which will be denoted by e.

2. Main results and terminologies

An element x in a normed algebra X is called a left (right) topological divisor of zero, briefly, a left (right) TDZ, provided there exists a sequence $y_n \in X$ of unit elements such that $xy_n \to 0$ ($y_n x \to 0$). It is merely called a TDZ if it is either a left or a right TDZ. It is a two-sided TDZ if it is both a left and a right TDZ.

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DEFINITION 2. 1. Let X be a normed algebra. For $x\in X$, we shall define three subsets $L(x)$, $R(x)$ and $LR(x)$ of C by

$$L(x)=\{z\in C : x-ez \text{ is a left TDZ}\},$$
$$R(x)=\{z\in C : x-ez \text{ is a right TDZ}\}, \text{ and}$$
$$LR(x)=\{z\in C : x-ez \text{ is a two-sided TDZ}\}.$$

LEMMA 2. 2[4][7]. *Let X be a normed algebra. For a fixed $x\in X$ let f_x and g_x be two functions on C defined by*

$$f_x(z)=\inf_{y\in X}\frac{\|(x-ez)y\|}{\|y\|} \quad \text{and} \quad g_x(z)=\inf_{y\in X}\frac{\|y(x-ez)\|}{\|y\|}.$$

Then (1) *f_x and g_x are continuous. In fact, we have*

$$|f_x(z)-f_x(u)|\leq|z-u| \quad \text{and} \quad |g_x(z)-g_x(u)|\leq|z-u|.$$

(2) *$z\in L(x)$ iff $f_x(z)=0$ and*

$z\in R(x)$ iff $g_x(z)=0$.

(3) *If $(x-ez)(x-eu)$ is a left (right) TDZ, then either z or $u\in L(x)(R(x))$. In fact, we have*

$$f_x(z)f_x(u)\leq\inf_{y\in X}\frac{\|(x-ez)(x-eu)y\|}{\|y\|}$$
$$g_x(z)g_x(u)\leq\inf_{y\in X}\frac{\|y(x-ez)(x-eu)\|}{\|y\|}.$$

DEFINITION 2. 3. A Banach algebra X is a Banach*-algebra if there is an involution $a\to a^*$ defined on X with the following properties:

(a) $(\alpha a+\beta b)^*=\bar{\alpha}a^*+\bar{\beta}b^*$,

(b) $a^{**}=a$,

(c) $(ab)^*=b^*a^*$ and

(d) $\|a^*\|=\|a\|$ for a, b in X; α, β in C.

In what follows unless exception is noted, X denotes a Banach*-algebra.

DEFINITION 2. 4. $x\in X$ is called a normalized topological divisor of zero, briefly, NTDZ, if there exists a sequence $y_n\in X$ of unit elements such that $xy_n\to 0$ and $x^*y_n\to 0$.

For a fixed $x\in X$, the set $\sigma_n(x)$ of all $z\in C$ such that $x-ez$ is a NTDZ is called the normal spectrum of x.

As usual $\sigma(x)$ denotes the spectrum of $x\in X$ and $\partial\sigma(x)$ its boundary.

THEOREM 2.5. *For* $x\in X$, $\sigma_n(x)\subset L(x)\cap R(x)\subset L(x)\cup R(x)\subset\sigma(x)$.

Proof. First we shall show that $\sigma_n(x)\subset L(x)\cap R(x)$. Let $z\in\sigma_n(x)$. Then $x-ez$ is an NTDZ. i.e., there exists a sequence $y_n\in X$ of unit elements such that $(x-ez)y_n\to 0$ and $(x-ez)^*y_n\to 0$. Since $((x-ez)^*y_n)^*=y_n^*(x-ez)$ and $\|y_n^*\|=\|y_n\|=1$ for all n. Hence $z\in L(x)\cap R(x)$. Finally to show the last inclusion, let $z\in L(x)\cup R(x)$. Then $x-ez$ is either left or right singular, and so $z\in\sigma(x)$.

COROLLARY 2.6. *For* $x\in X$ *and* $\lambda\in C$

(1) $\lambda\in\sigma_n(x)$ *iff* $P_x(\lambda)=0$, *where* $P_x(\lambda)=\inf\limits_{y\in X}\dfrac{\|(x-e\lambda)y\|+\|(x-e\lambda)^*y\|}{\|y\|}$.

(2) $\lambda\in LR(x)$ *iff* $f_x(\lambda)=g_x(\lambda)=0$ *iff* $q_x(\lambda)=0$,
where $q_x(\lambda)=\inf\limits_{y\in X}\dfrac{\|(x-e\lambda)y\|+\|y(x-e\lambda)\|}{\|y\|}$ *and* f_x *and* g_x *are as in Lemma* 2.2.

(3) *If* $f_x(\lambda)=0$ *or* $g_x(\lambda)=0$, *then* $\lambda\in\sigma(x)$.

Proof. (1) If $\lambda\in\sigma_n(x)$, then $x-\lambda e$ is an NTDZ. Hence there exists a sequence $y_n\in X$ such that $\|y_n\|=1$ and $(x-\lambda e)y_n\to 0$ and $(x-\lambda e)^*y_n\to 0$. Since

$$\inf_{y\in X}\frac{\|(x-\lambda e)y\|+\|(x-\lambda e)^*y\|}{\|y\|}\leq\inf_{\substack{y_n\in X\\(\|y_n\|=1)}}(\|(x-\lambda e)y_n\|+\|(x-\lambda e)^*y_n\|),$$

we have $P_x(\lambda)=0$.

Conversely, if $P_x(\lambda)=0$, there exists a sequence $y_n\in X$ such that $\|y_n\|=1$ and $\|(x-\lambda e)y_n\|+\|(x-\lambda e)^*y_n\|\to 0$. Hence $\lambda\in\sigma_n(x)$.

(2) is obvious.

(3) If $\lambda\in L(x)\cup R(x)$, then $\lambda\in\sigma(x)$ results from Lemma 2.2 and Theorem 2.5.

COROLLARY 2.7. *Let* X *be a* B^**-algebra.*

(1) *If* $x\in X$ *is a hyponormal element, i.e.,* $x^*x\geq xx^*$, *then* $\sigma_n(x)=L(x)\subset R(x)=\sigma(x)$.

(2) *If* $x\in X$ *is a normal element, i.e.,* $x^*x=xx^*$, *then* $\sigma_n(x)=L(x)=R(x)=\sigma(x)=LR(x)$.

Proof. (1) $x-\lambda e$ is a hyponormal iff x is a hyponormal. Thus in order to show that $\sigma_n(x)=L(x)$, it is sufficient to show that if $0\in L(x)$ then $0\in\sigma_n(x)$. Since $x^*x\geq xx^*$ and X is a B^*-algebra, $(xy_n)^*(xy_n)\geq(x^*y_n)^*(x^*y_n)$ for any bounded sequence $y_n\in X$. Then $\|xy_n\|^2\geq\|x^*y_n\|^2$, since $(x^*y_n)^*(x^*y_n)\geq 0$. Thus $xy_n\to 0$ implies $x^*y_n\to 0$; so $L(x)=\sigma_n(x)$. The last equality is well known, since X is a B^*-algebra.

(2) The equalities hold because of (1), since x is a normal and $R(x)=L(x)$.

DEFINITION 2.8. $x\in X$ is called a M-hyponormal element if there exists a number M such that $\|(x-\lambda e)^*y\|\leq M\|(x-\lambda e)y\|$ for all y in X and for all λ in C.

THEOREM 2.9. *Let X be a B^*-algebra. If $x\in X$ is an M-hyponormal element, then $Re\,\sigma(x)\subset\sigma(Re\,x)$, where $Re\,x=\frac{1}{2}(x+x^*)$.*

Proof. Let λ be in $\mathrm{Re}\,\sigma(x)$. Then there exists $z\in\partial\sigma(x)$ such that $\mathrm{Re}\,z=\lambda$. Since $\partial\sigma(x)\subseteq LR(x)=\sigma_n(x)=L(x)\subset R(x)=\sigma(x)$ by x is a M-hyponormal element, so $(\mathrm{Re}\,x-\lambda e)y_n=[\mathrm{Re}(x-z)]y_n=\frac{1}{2}[(x-ez)+(x-ez)^*]y_n\to 0$ and $\|y_n\|=1$ for all n. Hence $\lambda\in\sigma(\mathrm{Re}\,x)$.

THEOREM 2.10. *Let $x\in X$. Then $R(x)$, $\sigma_n(x)$ and $LR(x)$ are all compact subsets of $\sigma(x)$.*

Proof. To show that $R(x)$ is closed in $\sigma(x)$, we prove that if $z\notin R(x)$, then any $u\in C$ with $|z-u|<g_x(z)$ is not in $R(x)$, because $0<g_x(z)-|z-u|\leq g_x(u)$ by Lemma 2.2. It shows that $R(x)$ is closed in $\sigma(x)$ and hence a compact subset of C. Similarly, $L(x)$ is compact. As for $\sigma_n(x)$ and $LR(x)$, we have

$$\begin{aligned} q_x(z) &= \inf_{y\in X}\frac{\|(x-ze)y\|+\|y(x-ze)\|}{\|y\|} \\ &\leq \frac{\|(ue-ze)y+(\mathrm{x}-ue)y\|}{\|y\|}+\frac{\|y(ue-ze)+y(x-ue)\|}{\|y\|} \\ &\leq 2|z-u|+q_x(u) \end{aligned}$$

Similarly $|P_x(z)-P_x(u)|\leq 2|z-u|$. The same argument as for $R(x)$ shows that $\sigma_n(x)$ and $LR(x)$ are compact subsets of C.

COROLLARY 2.11. *For a fixed* $z\in C$, *the following four sets are closed in* X. $\{x : x-ze$ *is a left TDZ*$\}$, $\{x : x-ze$ *is a right TDZ*$\}$, $\{x : x-ze$ *is a NTDZ*$\}$ *and* $\{x : x-ze$ *is a two sided TDZ*$\}$.

Poof. Define $h_z(x)=\inf_{y\in X}\frac{\|(x-ze)y\|}{\|y\|}$. Then $x\in\{x : x-ze$ is a left TDZ$\}$ iff $h_z(x)=0$. Also, $h_z(x)\leq\|x-y\|+h_z(y)$ for y in X, and by the same argument as in theorem 2.12, $\{x : x-ze$ is a left TDZ$\}$ is closed in X. Let $l_z(x)=\inf_{y\in X}\frac{\|(x-ze)y\|+\|(x-ze)^*y\|}{\|y\|}$. Then $x\in\{x : x-ze$ is a NTDZ$\}$ iff $l_z(x)=0$. Also $l_z(x)\leq 2\|x-y\|+l_z(y)$ for y in X, hence $\{x : x-ze$ is a NTDZ$\}$ is closed in X. The proofs of other cases are similar to those of the previous ones and are omitted.

3. Applications of topological divisors of zero

Let A be an algebra; let $G_l=G_l(A)$ be the set of all left invertible elements of A and $G_r=G_r(A)$ be the set of all right invertible elements of A and we define $\sigma_l(x)=\{z\in C : x-ez\notin G_l\}$, $\sigma_r(x)=\{z\in C : x-ez\notin G_r\}$ and the two sided spectrum $\sigma_t(x)=\sigma_l(x)\cap\sigma_r(x)$. We also define the resolvent set $\rho(x)=\sigma(x)^c$, where $\sigma(x)^c$ is the complement of $\sigma(x)$, and $\rho_l(x)=\sigma_l(x)^c$, $\rho_r(x)=\sigma_r(x)^c$ and $\rho_t(x)=\sigma_t(x)^c=\rho_l(x)\cup\rho_r(x)$ are defined.

LEMMA 3.1. *Let* A *be a normed algebra with continuous inverse* [6]. *If* $x\in A$, *then* $\partial\sigma(x)\subset LR(x)$.

Proof. Since $\sigma(x)$ is a nonempty closed subset of C [6], we obtain that $\partial\sigma(x)$ is a nonempty set. Let $z\in\partial\sigma(x)$. Then $z\in\sigma(x)$ and there exists a sequence $\{z_n\}$ in $\sigma(x)^c$ such that $\lim_{n\to\infty} z_n=y$. Therefore $ze-x$ is not invertible but $\lim_{n\to\infty}(z_ne-x)=ze-x$. Thus $ze-x$ is the boundary of the invertible elements, so $ze-x$ is a two sided TDZ.

THEOREM 3.2. *Let* A *be a normed algebra with continuous inverse. If* $x\in A$, *then* $\sigma_t(x)$ *is a nonempty closed subset of* C.

Proof. Since $\partial\sigma(x)$ is a nonempty set, let $z\in\partial\sigma(x)$, then $x-ze$ is a two sided TDZ. So there exists a sequence y_n in A such that $\|y_n\|=1$, $(x-ze)y_n\to 0$ and $y_n(x-ze)\to 0$. Suppose that $a(x-ze)=e$. Then $1=$

$\|y_n\|=\|ey_n\|=\|a(x-ze)y_n\|\leq\|a\|\|(x-ze)y_n\|\to 0$. It is a contradiction. So $z\in\sigma_l(x)$. Similarly $z\in\sigma_r(x)$. Thus $\partial\sigma(x)\subset\sigma_l(x)\cap\sigma_r(x)=\sigma_t(x)$. Hence $\sigma_t(x)$ is a nonempty. Since $\sigma_t(x)\subset\sigma(x)$, to show that $\sigma_t(x)$ is closed, it is sufficient to show that $\rho_t(x)$ is open. This follows immediately, since $G_l(A)\cup G_r(A)$ is open.

COROLLARY 3.3. *If A is a Banach algebra and $x\in A$, then $\sigma_l(x)$ and $\sigma_r(x)$ are nonempty compact subset of C.*

Proof. Since $\sigma(x)$ is a compact subset of C, $\sigma_t(x)\subset\sigma_l(x)(\sigma_r(x))\subset\sigma(x)$ and $\sigma_l(x)^c=\rho_l(x)$ is open, these results are obvious.

COROLLARY 3.4. *If A is a Banach algebra in which every nonzero element is left invertible or right invertible, then A is (isometrically isomorphic to) the complex field C.*

Proof. Let $x\in A$ be arbitrary and $z\in\sigma_t(x)$. Then $ze-x\notin G_l(A)\cup G_r(A)$, this implies that $ze-x=0$; so $x=ze$.

Remark. This corollary improves the Gelfand-Mazur Theorem [8], but the proof given above is more elementary.

THEOREM 3.5. *Let A be a Banach algebra, and let $x\in A$. If $\sigma(x)^0=\phi$ (i.e., $\sigma(x)$ has empty interior), then $\sigma_t(x)=\sigma_l(x)=\sigma_r(x)=\sigma(x)$.*

Proof. Since $\sigma(x)=\partial\sigma(x)$, we must show that $\partial\sigma_l(x)\subset\sigma_r(x)$ and $\partial\sigma_r(x)\subset\sigma_l(x)$. First we prove that $\partial\sigma_l(x)\subset\partial_r(x)$. Let $z\in\partial\sigma_l(x)$. Then there exists a sequence $\{z_n\}$ in $\sigma_l(x)^c$ such that $z_n\to z$, but $z\in\sigma_l(x)$. If $z\notin\sigma_r(x)$, then $ze-x\in G_r(A)$. Since $G_r(A)$ is open and $z_ne-x\to ze-x$, so $z_ne-x\in G_r(A)$ for sufficiently many n; so $z_n\in\sigma_r(x)$. We may assume that $\{z_n\}\subset\rho_l(x)\cap\rho_r(x)=\rho(x)$ for all n. But then $\{z_n\}\subset\rho(x)$, $z\in\sigma(x)$ and $z_n\to z$. So $z\in\partial\sigma(x)\subset\sigma_r(x)$. It is a contradiction. So $\partial\sigma_l(x)\subset\sigma_r(x)\subset\sigma(x)$. Similarly we have $\partial\sigma_r(x)\subset\sigma_l(x)\subset\sigma(x)$. Consequently $\partial\sigma_l(x)\subset\sigma_r(x)\cap\sigma_l(x)=\sigma_t(x)\subset\sigma(x)=\partial\sigma(x)$. Thus $\sigma(x)=\sigma_t(x)$ and the another equalities are analogous.

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