Comm. Korean Math. Soc. 3(1988), No. 2, pp. 179~187

THE REPRODUCING KERNEL OF HILBERT SPACE OF YEH-WIENER PROCESS

HONG-TAEK HWANG AND HEE-SOOK PARK

1. Introduction

Let $Q=[0,p]\times[0,q]$, where p and q are some fixed positive real numbers. Let $C_2[Q]$, called *Yeh-Wiener space*, denote the function space $\{x(\cdot,\cdot)\mid x(s,0)=x(0,t)=0,\ x(s,t)$ is a real valued continuous function on $Q\}$ with the uniform norm $\|x\|=\max_{(s,t)\in Q}|x(s,t)|$. Let J be the algebra of all subsets of $C_2[Q]$ of the form

$$I=\{x\in C_2[Q]\mid (x(s_1,t_1),\cdots,x(s_m,t_m))\in E\}$$

where m and n are any positive integers, $0=s_0<s_1<\cdots<s_m=p$, $0=t_0<t_1<\cdots<t_n=q$ and E is an arbitrary Lebesgue measurable set in the mn-dimensional Euclidean space R^{mn}. Let $(C_2[Q],\mathscr{Y},m_y)$, called the *Yeh-Wiener measure space*, denote the complete probability space where $\mathscr{Y}$ is the σ-algebra of Caratheodory measurable subsets of $C_2[Q]$ with respect to the outer measure induced by the probability measure m_y on the algebra $\mathscr{J}$ defined for $I\in\mathscr{J}$ by

$$m_y(I)=\prod_{j=1}^{m}\prod_{k=1}^{n}\{2\pi(s_j-s_{j-1})(t_k-t_{k-1})\}^{-1/2}\cdot$$

$$\int_E\exp\left\{-\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{(u_{j,k}-u_{j-1,k}-u_{j,k-1}+u_{j-1,k-2})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}$$
$$du_{11}\cdots du_{mn}$$

where $u_{0,k}=u_{j,0}=u_{0,0}=0$ $(j=1,2,\cdots,m,\ k=1,2,\cdots,n)$. (see [3]).

Let Y be a real valued function on $Q\times C_2[Q]$ defined by

$$Y((s,t),x)=x(s,t)\ \text{ for }\ ((s,t),x)\in Q\times C_2[Q].$$

Received July 2, 1988.

Then Y is a measurable stochastic process in the probability space $(C_2[Q], \mathscr{Y}, m_y)$ and the domain Q in which the space of sample functions $Y(\cdot, x)$, $x \in C_2[Q]$, coincides with the sample space $C_2[Q]$. This stochastic process will be referred to as the *Yeh-Wiener process* on the domain Q. It is shown from the definition of $(C_2[Q], \mathscr{Y}, m_y)$ that $Y((s,t), \cdot) \sim N(0, st)$ (i.e. normally distributed with mean 0 and variance st) and covariance of *Yeh-Wiener process*, $E^y(Y((s,t), \cdot) \cdot Y((u,v), \cdot)) = \min\{s,u\} \cdot \min\{t,v\}$ for every (s,t), $(u,v) \in Q$.

In this paper, we shall show that the function Γ defined on Q by

$$\begin{aligned}\Gamma((s,t), \ (u,v)) &= E^y(Y(s,t), \cdot)\, Y((u,v), \cdot) \\ &= \min\{s,u\} \min\{t,v\}\end{aligned}$$

for every (s,t), $(u,v) \in Q$, is a covariance kernel on Q and then find a reproducing kernel Hilbert spae (i.e. *Yeh-Wiener process*) of the covariance kernel Γ on Q. We next study some linear operator and the reproducing kernel of Hilbert space.

2. The reproducing kernel of Hilbert space of Yeh-Wiener process

In this section we find the reproducing kernel of Hilbert space of a covariance kernel defined on a rectangula $[0,p] \times [0,q]$ and then study some linear operators in the reproducing kernel of Hilbert space.

DEFINITION 2.1. Let T be a set and $\Gamma : T \times T \to \boldsymbol{R}$ be a real valued mapping satisfying

(1) given $s, t \in T$, $\Gamma(s,t) = \Gamma(t,s)$, and

(2) Γ is nonnegative definite i.e., for given $t_j \in T$, $a_j \in C$

$$j = 1, 2, \cdots, n, \quad \sum_{j,k} \Gamma(t_j, t_k) a_j \bar{a}_k \geq 0.$$

Then $\Gamma(\cdot, \cdot)$ is called a *covariance kernel* on $T \times T$.

Let T be a separable metric space, and let $\Gamma(\cdot, \cdot)$ be a covariance kernel on $T \times T$. Define

$$V = \{\sum_{j=1}^{n} a_j \Gamma(t_j, \cdot) : a_j \in \boldsymbol{R}, \ t_j \in T, \ j = 1, 2, \cdots, n\}$$

and the inner product $(\cdot, \cdot) : V \times V \to \boldsymbol{R}$ by

$$(\sum_{j=1}^{n} a_j \Gamma(t_j, \cdot), \sum_{k=1}^{m} b_k \Gamma(s_k, \cdot)) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \Gamma(t_j, s_k).$$

Then it is easy to see that $(\cdot, \cdot)$ is an inner product on V. If $\{f_n\}$ is a sequence in V, then

$$|f_n(t)-f_m(t)|^2=|(f_n-f_m, \Gamma(t, \cdot))|^2$$
$$\leqslant (f_n-f_m, f_n-f_m)(\Gamma(t, \cdot), \Gamma(t, \cdot))$$
$$=\|f_n-f_m\|^2\Gamma(t,t),$$

where $(f,f)=\|f\|^2$, $f\in V$. It follows that if $\{f_n\}$ is a Cauchy sequence with respect to the induced norm $\|\cdot\|$, then it is a pointwise Cauchy sequence. Let $H(\Gamma)$ denote the completion of $(V, \|\cdot\|)$. This space $H(\Gamma)$ will be called the reproducing kernel Hilbert space of the covariance kernel $\Gamma(\cdot, \cdot)$ on $T\times T$.

THEOREM 2. 2. [1] *Let $\Gamma(\cdot, \cdot)$ be a covariance kernel on $T\times T$. Let $H=\{f: f: T\to R$ is a function$\}$ be a Hilbert space with the inner product $(\cdot, \cdot)_H$. Suppose*

(1) $\Gamma(t, \cdot)\in H$, $t\in T$ and

(2) $(f(\cdot), \Gamma(t, \cdot))_H=f(t)$, $t\in T$, $f\in H$.

Then H is equal to $H(\Gamma)$ as Hilbert spaces.

THEOREM 2. 3. *For (s,t), $(u,v)\in[0,p]\times[0,q]$, let $\Gamma((s,t), (u,v))=\min\{s,u\}\cdot\min\{t,v\}$. Then*

(1) Γ is a covariance kernel on $[0,p]\times[0,q]$.

(2) $H(\Gamma)=\{f: f(s,t)=\int_0^t\int_0^s D^2f(u,v)\, dudv$ *and*

$$\int_0^q\int_0^p[D^2f(s,t)]^2\, dsdt<\infty\}$$

where $D^2f(s,t)=\frac{\partial^2}{\partial_s\partial_t}f(s,t)$.

Proof. (1) For (s,t), $(u,v)\in[0,p]\times[0,q]$, we have $\Gamma((s,t), (u,v))=\min\{s,u\}\cdot\min\{t,v\}=\Gamma((u,v), (s,t))$. Let $(s_i, t_i)\in[0,p]\times[0,q]$, $c_i\in C$, $i=1,2,\cdots,n$. Then

$$\sum_{i,j}^{n}c_i\bar{c}_j\Gamma((s_i,t_i), (s_j,t_j))=\sum_{i,j}^{n}c_i\bar{c}_j(s_i\wedge s_j)(t_i\wedge t_j)$$

$$=\int_0^q\int_0^p\sum_{i,j}^{n}c_i\bar{c}_j\chi_{[0,s_i]}(s)\chi_{[0,s_j]}(s)\chi_{[0,t_i]}(t)\chi_{[0,t_j]}(t)dsdt$$

$$=\int_0^q\int_0^p\sum_{i,j}^{n}c_i\bar{c}_j\frac{\partial^2\Gamma((s_i,t_i), (s,t))}{\partial s\partial t}\cdot\frac{\partial^2\Gamma((s_j,t_j), (s,t))}{\partial s\partial t}dsdt$$

$$=\int_0^q\int_0^p \left| \sum_i^n c_i \frac{\partial^2 \Gamma((s_i,t_i),\ (s,t))}{\partial s \partial t} \right|^2 ds dt \geqq 0.$$

Hence $\Gamma(\cdot,\cdot)$ is a covariance kernel on $T\times T$.

(2) For any $(s,t)\in[0,p]\times[0,q]$,

$$\frac{\partial}{\partial x}\Gamma((s,t),\ (x,y))=(t\wedge y)\chi_{[0,s]}(x),\ \ x\neq s$$

and

$$\frac{\partial}{\partial y}\Gamma((s,t),\ (x,y))=(s\wedge x)\chi_{[0,t]}(y),\ \ y\neq t.$$

It follows that

$$\begin{aligned}\Gamma((s,t),\ (x,y)) &= (s\wedge x)(t\wedge y)\\ &= \int_0^y\int_0^x D^2\Gamma((s,t),\ (u,v))\ dudv.\\ \|\Gamma((s,t),\ (x,y))\|^2 &= \Gamma((s,t),\ (s,t))=st\\ &= \int_0^q\int_0^p [D^2\Gamma((s,t),\ (u,v))]^2\ dudv.\end{aligned}$$

Let

$$f(x,y)=\sum_{j=1}^n c_j\Gamma((s_j,t_j),\ (x,y)).$$

Then we have

$$\begin{aligned}\|f\|^2 &= \sum_{j,k} c_jc_k(s_j\wedge s_k)(t_j\wedge t_k)\\ &= \int_0^q\int_0^p \sum_{j,k} c_jc_k[D^2\Gamma((s_j,t_j),\ (x,y))][D^2\Gamma((s_k,t_k),(x,y))]\ dxdy\\ &= \int_0^q\int_0^p [D^2 f(x,y)]^2\ dxdy.\end{aligned}$$

Let

$$H=\{f: f(u,v)=\int_0^v\int_0^u D^2 f(x,y)\ dxdy,\ \int_0^q\int_0^p [D^2 f(x,y)]^2\ dxdy<\infty\}.$$

Now, for $f,g\in H$, define

$$(f,g)=\int_0^q\int_0^p D^2f(x,y)\ D^2g(x,y)\ dxdy.$$

Then it is easy to see that $(\cdot,\cdot)$ is an inner product on H and that H is a Hilbert space with respect to $(\cdot,\cdot)$; for $(s,t)\in[0,p]\times[0,q]$, clearly $\Gamma((s,t),\ (x,y))\in H$ and

$$\begin{aligned}
&(f(\cdot,\cdot),\ \Gamma((s,t),\ (\cdot,\cdot)))\\
&=\int_0^q\int_0^p D^2f(x,y)D^2\Gamma((s,t),(x,y))\ dxdy\\
&=\int_0^q\int_0^p D^2f(x,y)\chi_{[0,s]}(x)\chi_{[0,t]}(y)\ dxdy\\
&=\int_0^t\int_0^s[D^2f(x,y)]\ dxdy\\
&=f(s,t).
\end{aligned}$$

Therefore by Theorem 2.2, $H(\Gamma)=H$.

The following well-known fact is a version of integration by parts formula for functions of two variables.

LEMMA 2.4. *Let f and g be real valued functions on $[0,p]\times[0,q]$. If f and g are in $H(\Gamma)$, then*

$$\begin{aligned}
\int_0^q\int_0^p[D^2f]g\,dsdt=f(p,q)-\int_0^q\frac{\partial g}{\partial t}(p,t)f(p,t)dt\\
-\int_0^p\frac{\partial g}{\partial s}(s,q)f(s,q)ds+\int_0^q\int_0^p[D^2g]f\,dsdt.
\end{aligned}$$

THEOREM 2.5. *Let Γ and $H(\Gamma)$ be as in Theorem 2.3. Define an operator $S:H(\Gamma)\to H(\Gamma)$ by*

$$Sf(s,t)=\int_0^t\int_0^s f(u,v)\ dudv.$$

Then S is a bounded linear operator and the adjoint operator S^* of S given by

$$\begin{aligned}
S^*f(s,t)=stf(p,q)-s\int_0^t f(p,v)dv-t\int_0^s f(u,q)du\\
+\int_0^t\int_0^s f(u,v)\ dudv.
\end{aligned}$$

Proof. For $f\in H(\Gamma)$, then

$$\int_0^v\int_0^u D^2Sf(s,t)dsdt=\int_0^v\int_0^u\frac{\partial^2}{\partial s\partial t}\left[\int_0^t\int_0^s f(x,y)dxdy\right]dsdt$$
$$=\int_0^v\int_0^u f(s,t)dsdt.$$

Then S is well-defined. For $\alpha, \beta\in \boldsymbol{R}$ and $f, g\in H(\Gamma)$, we have

$$S(\alpha f+\beta g)(s,t)=\int_0^t\int_0^s(\alpha f+\beta g)(u,v)dudv$$
$$=\alpha\int_0^t\int_0^s f(u,v)dudv+\beta\int_0^t\int_0^s g(u,v)dudv$$
$$=\alpha(Sf)(s,t)+\beta(Sg)(s,t).$$

Now we note that

$$\|Sf\|^2=\int_0^q\int_0^p[D^2Sf(s,t)]^2dsdt=\int_0^q\int_0^p[f(s,t)]^2dsdt.$$

Hence by Hölder inequality,

$$\|Sf\|^2\geqslant\int_0^q\int_0^p D^2f(u,v)dudv\int_0^q\int_0^p dsdt$$
$$=\|f\|_{pq}{}^2.$$

Hence S is a bounded linear operator on $H(\Gamma)$. By Lemma 2.4,

$$(f,S_g)=\int_0^q\int_0^p D^2f\cdot gdsdt.$$
$$=f(p,q)\int_0^q\int_0^p D^2gdsdt-\int_0^q\left[\frac{\partial g(p,t)}{\partial t}\right]f(p,t)dt$$
$$-\int_0^p\left[\frac{\partial g(s,q)}{\partial s}\right]f(s,q)ds+\int_0^q\int_0^p D^2g\cdot fdsdt.$$

Note that

$$\int_0^q\left[\frac{\partial g(p,t)}{\partial t}\right]f(p,t)dt=\int_0^q\int_0^p D^2g(s,t)f(p,t)dsdt$$

and

$$\int_0^p\left[\frac{\partial g(s,q)}{\partial s}\right]f(s,q)ds=\int_0^p\int_0^q\left[\frac{\partial^2 g(s,t)}{\partial t\partial s}\right]f(s,q)dtds$$
$$=\int_0^q\int_0^p\left[\frac{\partial^2 g(s,t)}{\partial s\partial t}\right]f(s,q)dsdt$$

$$\int_0^q\int_0^p D^2f\cdot g dsdt=\int_0^q\int_0^p D^2g[f(p,q)-f(p,t)-f(s,q)-f(s,t)]dsdt.$$

From the relation $(f, S_g)=(S^*f, g)$, we obtain

$$\int_0^q\int_0^p D^2g[f(p,q)-f(p,t)-f(s,q)+f(s,t)]dsdt$$
$$=\int_0^q\int_0^p D^2(S^*f)D^2g dsdt$$

so that we have

$$D^2(S^*f)=f(p,q)-f(p,t)-f(s,q)+f(s,t).$$

Hence the adjoint operator S^* of S is given by

$$\begin{aligned}S^*f(u,v)&=\int_0^v\int_0^u D^2(S^*f)dsdt\\&=\int_0^v\int_0^u[f(p,q)-f(p,t)-f(s,q)+f(s,t)]dsdt\\&=uvf(p,q)-u\int_0^v f(p,t)dt-v\int_0^u f(s,q)ds+\int_0^v\int_0^u f(s,t)dsdt.\end{aligned}$$

THEOREM 2.6. *Let* $A=S^*S$. *Then* A *is a self-adjoint operator on* $H(\Gamma)$ *defined by*

$$Af(s,t)=\int_0^q\int_0^p \min\{s,u\}\min\{t,v\}f(u,v)dudv.$$

Proof. For any $f\in H(\Gamma)$, we have

$$\begin{aligned}Af(s,t)&=S^*Sf(s,t)\\&=S^*\Big[\int_0^t\int_0^s f(u,v)dudv\Big]\\&=\int_0^t\int_0^s\Big[\int_0^q\int_0^p f(u,v)dudv-\int_0^y\int_0^p f(u,,v)dudv\\&\qquad-\int_0^q\int_0^x f(u,v)dudv+\int_0^y\int_0^x f(u,v)dudv\Big]dxdy\\&=\int_0^t\int_0^s\Big[\int_y^q\int_0^p f(u,v)dudv-\int_y^q\int_0^x f(u,v)dudv\Big]dxdy\\&=\int_0^t\int_0^s\Big[\int_y^q\int_x^p f(u,v)dudv\Big]dxdy.\end{aligned}$$

Now put

$$g(x,y)=\int_y^q\int_x^p f(u,v)\,dudv \text{ and } h(x,y)=xy.$$

Then by Lemma 2.4, we have

$$\begin{aligned}
Af(s,t)&=\int_0^t\int_0^s g(x,y)\,dxdy\\
&=stg(s,t)-\int_0^g\Big[\frac{\partial g(p,y)}{\partial y}\Big]h(p,y)\,dy\\
&\quad-\int_0^p\Big[\frac{\partial g(x,q)}{\partial x}\Big]h(x,q)\,dx+\int_0^t\int_0^s D^2g\cdot h\,dxdy\\
&=st\int_t^q\int_s^p f(u,v)\,dudv-\int_0^q\Big[\frac{\partial}{\partial y}\int_y^q\int_p^p f(u,v)\,dudv\Big]py\,dy\\
&\quad-\int_0^p\Big[\frac{\partial}{\partial x}\int_q^q\int_x^p f(u,v)\,dudv\Big]qx\,dx\\
&\quad+\int_0^t\int_0^s\Big[\frac{\partial^2}{\partial x\partial y}\Big(\int_x^p\int_y^q f(u,v)\,dudv\Big]xy\,dxdy\\
&=st\int_t^q\int_s^p f(u,v)\,dudv+\int_0^t\int_0^s xyf(x,y)\,dxdy\\
&=\int_0^q\int_0^p \min\{s,x\}\min\{t,y\}f(x,y)\,dxdy.
\end{aligned}$$

COROLLARY 2.7. *Let A be the operator as in Theorem 2.6. Then for $f,g\in H(\Gamma)$,*

$$(f,A_g)=\int_0^q\int_0^p f(u,v)g(u,v)\,dudv.$$

Consequently, A is positive definite; i.e. $(Af,f)\geqslant 0$ for $f\in H(\Gamma)$.

Proof. Since $A=S^*S$, *we have*

$$\begin{aligned}
(f,Ag)&=(Sf,Sg)\\
&=\int_0^q\int_0^p D^2(Sf)D^2(Sg)\,dsdt\\
&=\int_0^q\int_0^p f(s,t)g(s,t)\,dsdt.
\end{aligned}$$

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Kum-Oh Institute of Technology
Kumi 730-070, Korea
and
Sogang University
Seoul 121-742, Korea