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NONLINEAR ERGODIC THEOREMS FOR LIPSCHITZIAN MAPPINGS IN HILBERT SPACES

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1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and C a nonempty closed convex subset of H. A mapping $T: C\to C$ is called Lipschitzian on C([2]) if for each $n\geq 1$ there exists a positive real number k_n such that

$$\|T^n x - T^n y\|\leq k_n\|x-y\|$$

for all $x, y\in C$. A Lipschitzian mapping is nonexpansive if $k_n=1$ for all $n\geq 1$ and asymptotically nonexpansive if $\lim_{n\to\infty} k_n=1$ respectively.

The fixed point set of T is defined by

$$F(T)=\{x\in C: Tx=x\}.$$

The first nonlinear ergodic theorem for nonexpansive mappings was established by Baillon ([1]): Let C be a bounded closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x\in C$, the Cesàro mean

$$S_n x=\frac{1}{n}\sum_{k=0}^{n-1}T^k x$$

converges weakly as $n\to\infty$ to a fixed point y of T.

A corresponding result for an asymptotically nonexpansive mapping T on C was given by Hirano and Takahashi ([4]). In this case, putting $y=Px$ for each $x\in C$, P is a nonexpansive retraction of C onto

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$F(T)$ such that $PT=TP=P$ and $Px\in\overline{\text{conv}}\ \{T^n x : n=0, 1, 2 \cdots\}$ for each $x\in C$.

In this paper, we prove the existence of a nonexpansive retraction (ergodic retraction) for a Lipschitzian mapping.

Some rudimental definitions and notations in the Hilbert spaces are necessary for the proof of the main theorem of this paper. Let $D\subset H$. We denote by $\bar{D}$ the closure of D, by conv D the convex hull of D. Let $\{x_n\}$ be a bounded sequence in C. Then we define

$$A(y)=\lim_{n\to\infty}\sup\ \|x_n-y\|^2,\quad y\in C.$$

It is well known there exists a unique element a in C with

$$A(a)=\inf_{y\in C}\ A(y),$$

called the asymptotic center of $\{x_n\}$ in C. We denote by $AC(\{x_n\})$ the asymptotic center of $\{x_n\}$. This definition is due to Lim and Pazy ([6], [7]). We also know that if $\{x_n\}\subset C$ and $\{x_n\}$ converges weakly to $y\in C$ then $y=AC(\{x_n\})$ ([3]). Let T be a Lipschitzian mapping. Then Q is said to be a metric projection of H onto $F(T)$ if for every $x\in H$

$$\langle x-Qx,\ Qx-u\rangle\geq 0$$

for all $u\in F(T)$. The metric projection Q is nonexpansive ([8]).

2. Nolinenar ergodic theorem

Before proving the nonlinear ergodic theorem we prove the following crucial propositions and lemma.

PROPOSITION 1. *Let C be a closed convex subset of a real Hilbert space H, T a Lipschitzian mapping on C into itself with $\lim_{n\to\infty}\sup k_n\leq 1$ and $\{T^n x\}$ a bounded set for each $x\in C$. Then $F(T)$ is a nonempty, closed convex subset of C.*

Proof. By [3] and [5], $F(T)$ is nonempty. Closedness of $F(T)$ is obvious. To show convexity, it is sufficient to prove that $z=(x+y)/2\in F(T)$ for all $x, y\in F(T)$. Since

$$\|T^n z - x\| = \|T^n z - T^n x\| \le k_n \|z - x\|$$
$$= \frac{1}{2} k_n \|x - y\|$$

and

$$\|T^n z - y\| = \|T^n z - T^n y\| \le k_n \|z - y\|$$
$$= \frac{1}{2} k_n \|x - y\|,$$

we have

$$\|T^n z - z\|^2 = \frac{1}{2}\|T^n z - x\|^2 + \frac{1}{2}\|T^n z - y\|^2 - \frac{1}{4}\|x - y\|^2,$$
$$\le \frac{1}{4}(k_n{}^2 - 1)\|x - y\|^2$$

and hence

$$\lim_{n\to\infty} \|T^n z - z\|^2 = 0.$$

Therefore we obtain

$$z = \lim_{n\to\infty} T^n z = \lim_{n\to\infty} T^{n+1} z = T(\lim_{n\to\infty} T^n z)$$
$$= Tz.$$

The proof is completed.

LEMMA 2. *Let C and T satisfy the assumptions as in Proposition* 1. *If $a = AC\{T^n x\}$, then we have*

$$A(a) + \|a - y\|^2 \le A(y)$$

for every $y \in C$.

Proof. Since for all $y \in C$, $0 < \lambda < 1$,

$$\|T^n x - (\lambda y + (1-\lambda)a)\|^2 = (1-\lambda)\|T^n x - a\|^2 + \lambda \|T^n x - y\|^2$$
$$- \lambda(1-\lambda)\|a - y\|^2,$$
$$A(a) \le A(\lambda y + (1-\lambda)a)$$
$$= \lim_{n\to\infty} \sup \|T^n x - (\lambda y + (1-\lambda)a)\|^2$$

$$=\limsup_{n\to\infty}\{(1-\lambda)\|T^nx-a\|^2+\lambda\|T^nx-y\|^2-\lambda(1-\lambda)\|a-y\|^2\}$$

$$\leq(1-\lambda)A(a)+\lambda A(y)-\lambda(1-\lambda)\|a-y\|^2$$

and hence

$$A(a)\leq A(y)-(1-\lambda)\|a-y\|^2.$$

Letting $\lambda\to 0^+$, we have

$$A(a)+\|a-y\|^2\leq A(y).$$

LEMMA 3. *Let C and T satisfy the same assumptions as in Proposition 1. Then we have*

$$AC(\{T^nx\})\in F(T)$$

for all $x\in C$.

Proof. Let $a=AC(\{T^nx\})$ for all $x\in C$ and Q the metric projection on $F(T)$. Then we have

$$\langle T^nx-Qa,\ Qa-a\rangle\geq 0.$$

Since

$$\|T^nx-a\|^2=\|T^nx-Qa\|^2+\|Qa-a\|^2+2\langle T^nx-Qa,\ Qa-a\rangle,$$

$$\limsup_{n\to\infty}\|T^nx-a\|^2\geq\limsup_{n\to\infty}\|T^nx-Qa\|^2+\|Qa-a\|^2.$$

Therefore we have

$$A(a)\geq A(Qa)+\|Qa-a\|.$$

Combining this and Lemma 2, we obtain $a=Qa\in F(T)$.

PROPOSITION 4. *Let C and T satisfy the same assumptions as in Propositions 1. If Q is the metric projection of C onto $F(T)$. Then for each $x\in C$, $\{QT^nx\}$ converges to the asymptotic center of $\{T^nx\}$ in $F(T)$.*

Proof. Since $\{T^nx\}$ is bounded for all $x\in C$, there exists a unique asymptotic center a of $\{T^nx\}$ in $F(T)$. Then we have

$$\begin{aligned}A(QT^n x) &\leq \sup_{m\geq p+n} \|T^m x - QT^n x\|^2 \\ &= \sup_{m\geq p} \|T^{m+n} x - QT^n x\|^2 \\ &= \sup_{m\geq p} \|T^m T^n x - T^m QT^n x\|^2 \\ &\leq (\sup_{m\geq p} k_m)^2 \|T^n x - QT^n x\|^2 \\ &\leq (\sup_{m\geq p} k_m)^2 \|T^n x - a\|^2\end{aligned}$$

for all m, $p=0, 1, 2, \cdots$.

Hence

$$\begin{aligned}A(QT^n x) &\leq (\limsup_{p\to\infty} k_p)^2 \|T^n x - a\|^2 \\ &\leq \|T^n x - a\|^2\end{aligned}$$

for all $n=0, 1, 2, \cdots$. Therefore we have

$$\begin{aligned}\limsup_{n\to\infty} A(QT^n x) &\leq \limsup_{n\to\infty} \|T^n x - a\|^2 \\ &= A(a).\end{aligned}$$

On the other hand, from Lemma 2, we have

$$A(a) + \|QT^n x - a\|^2 \leq A(QT^n x)$$

and hence

$$\begin{aligned}A(a) + \limsup_{n\to\infty} \|QT^n x - a\|^2 &\leq \limsup_{n\to\infty} A(QT^n x) \\ &\leq A(a).\end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty} \|QT^n x - a\|^2 = 0$$

and this implies $\{QT^n x\}$ converges to a in $F(T)$.

Now, we prove a key proposition which play a crucial role in the proof of our main theorem in this paper.

PROPOSITION 5. *Let C and T satisfy the same assumptions as in Proposition 1. Then the set*

$$\bigcap_m \overline{conv}\{T^n x : n \geq m\} \cap F(T)$$

contains at most one point.

If moreover

$$\bigcap_m \overline{conv}\{T^n x : n \geq m\} \cap F(T) \neq \phi,$$

then its element is the asymptotic center of $\{T^n x\}$.

Proof. Let $x \in C$ and $z \in F(T)$. Then for each m, $p = 0, 1, 2 \cdots$,

$$\begin{aligned}\lim_{n\to\infty} \sup \|T^n x - z\|^2 &\leq \sup_{m \geq n+p} \|T^m x - z\|^2 \\ &\leq \sup_{m \geq n} \|T^{m+p} x - z\|^2 \\ &= \sup_{m \geq n} \|T^{m+p} x - T^m z\|^2 \\ &\leq (\sup_{m \geq n} k_m)^2 \|T^p x - z\|^2.\end{aligned}$$

Hence we have

$$\begin{aligned}\lim_{n\to\infty} \sup \|T^n x - z\|^2 &\leq (\lim_{n\to\infty} \sup k_n)^2 \|T^p x - z\|^2 \\ &\leq \|T^p x - z\|^2\end{aligned}$$

for all $p = 0, 1, 2, \cdots$ and so

$$\lim_{p\to\infty} \sup \|T^p x - z\|^2 \leq \lim_{p\to\infty} \inf \|T^p x - z\|^2.$$

Therefore, there exists the limit of $\{\|T^n x - z\|\}$ as $n \to \infty$. Let a be the unique asymptotic center of $\{T^n x\}$ in $F(T)$ and $u \in \bigcap_m \overline{\text{conv}}\{T^n x : n \geq m\} \cap F(T)$.

Since

$$\begin{aligned}&\|a - u\|^2 + 2 \lim_{n\to\infty} \langle a - u,\ T^n x - a\rangle \\ &= \lim_{n\to\infty} \|T^n x - u\|^2 - \lim_{n\to\infty} \|T^n x - a\|^2 \\ &= A(u) - A(a) \\ &\geq 0,\end{aligned}$$

$$2\ \lim_{n\to\infty}\langle a-u,\ T^n x-a\rangle > -\|a-u\|^2-\varepsilon,$$

for all $\varepsilon>0$. Therefore there exists $n_0\geq 0$ such that

$$2\langle a-u,\ T^n x-a\rangle > -\|a-u\|^2-\varepsilon$$

for all $n\geq n_0$. Since $u\in\overline{\text{conv}}\ \{T^n x : n\geq n_0\}$, we have

$$2\langle a-u,\ u-a\rangle > -\|a-u\|^2-\varepsilon$$

and hence $\|a-u\|^2<\varepsilon$. Since ε is arbitrary, we have $u=a$, From the uniqueness of the asymptotic center of $\{T^n x\}$,

$$\bigcap_m \overline{\text{conv}}\{T^n x : n\geq m\}\cap F(T)$$

contains at most one point and its element is the asymptotic center of $\{T^n x\}$.

THEOREM 6. *Let C be a closed convex subset of a real Hilbert space H and T a Lipschitzian mapping on C into itself with* $\lim_{n\to\infty}\sup k_n\leq 1$. *If $F(T)$ is nonempty, then the following statements are equivalent:*

(1) $\bigcap_m \overline{conv}\{T^n x : n\geq m\}\cap F(T)\neq\phi$ *for each* $x\in C$,

(2) *There exists a nonexpansive retraction P of C onto $F(T)$ such that $PT=TP=P$ and $Px\in\overline{conv}\{T^n x : n=0, 1, 2, \cdots\}$ for every $x\in C$.*

Proof. (2)⇒(1). Let $x\in C$. Then $Px\in F(T)$. Furthermore, since for all $m=0, 1, 2, \cdots$,

$$\begin{aligned} Px&=PTx=PTTx=\cdots=PT^{m-1}x\\ &=PT^m x\\ &\in\overline{\text{conv}}\{T^{n+m}x : n=0, 1, 2, \cdots\}\\ &\subset\overline{\text{conv}}\{T^n x : n\geq m\}, \end{aligned}$$

$Px\in\bigcap_m\overline{\text{conv}}\{T^n x : n\geq m\}$. Hence we have

$$\bigcap_m\overline{\text{conv}}\{T^n x : n\geq m\}\cap F(T)\neq\phi.$$

(1)⇒(2). Let Q be a metric projection of C onte $F(T)$. Then by

Proposition 4, $\{QT^n x\}$ converges to the asymptotic center of $\{T^n x\}$ in $F(T)$. Let $Px = \lim_{n\to\infty} QT^n x$ for each $x \in C$. Then we have

$$\begin{aligned} TPx &= T(\lim_{n\to\infty} QT^n x) = \lim_{n\to\infty}(TQT^n x) \\ &= \lim_{n\to\infty}(QT^n x) = Px \end{aligned}$$

and

$$PTx = \lim_{n\to\infty} QT^{n+1}x = \lim_{n\to\infty} QT^n x = Px.$$

Since $\{Px\} = \bigcap_m \overline{\text{conv}}\{T^n x : n \geq m\} \cap F(T)$ by Proposition 5, it is true that

$$Px \in \overline{\text{conv}}\{T^n x : n = 0, 1, 2, \cdots\}$$

for all $x \in C$. Since Q is nonexpansive, we have

$$\begin{aligned} \|Px - Py\| &= \lim_{n\to\infty} \|QT^n x - QT^n y\| \\ &= \limsup_{n\to\infty} \|QT^n x - QT^n y\| \\ &\leq \limsup_{n\to\infty} \|T^n x - T^n y\| \\ &\leq (\limsup_{n\to\infty} k_n)\|x - y\| \\ &\leq \|x - y\|. \end{aligned}$$

Hence P is nonexpansive. Furthermore,

$$\begin{aligned} P^2 x &= \lim_{n\to\infty} QT^n Px = \lim_{n\to\infty} T^n Px \\ &= \lim_{n\to\infty} T^n(\lim_{m\to\infty} QT^m x) \\ &= \lim_{\substack{n\to\infty \\ m\to\infty}} (T^n QT^m x) \\ &= \lim_{m\to\infty} (QT^m x) \\ &= Px. \end{aligned}$$

This completes the proof.

REMARK. In Theorem 6, we know that the Cesàro mean

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to Px in $F(T)$.

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