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NOTES ON QUASI-ANALYTIC FUNCTIONS

S.S. KIM AND K.H. KWON

0. Introduction

Let Ω be an open set in R^1, which is assumed to be connected for simplicity. A function $f\in C^\infty(\Omega)$ is (real-) analytic if for any point x_0 in Ω, Taylor series expansion of f at x_0 converges to f in a neighborhood of x_0. Let us denote by $A(\Omega)$ the class of analytic functions in Ω. Analytic functions have the uniqueness property: if $f\in A(\Omega)$ is such that $f^{(n)}(x_0)=0$, $n\geq 0$, at some point $x_0\in\Omega$, then f must vanish identically in Ω. Among other things, it implies that there is no non-trivial function in $A(\Omega)$ with compact support so that we cannot localize the analysis in analytic category. Certainly, the uniqueness property does not hold in general in C^∞-category. However, there are subclasses of $C^\infty(\Omega)$ other than $A(\Omega)$ which have the uniqueness property. They are quasi-analytic classes $C^M(\Omega)$ (cf. Definition 1.1 and Theorem 1.1), where $M=(M_n)_0^\infty$ is a sequence of positive numbers satisfying Denjoy-Carleman condition. In this note, we shall first give a simple proof of characterization of quasi-analytic classes $C^M(\Omega)$ based on the existence of cut-off functions and then free the notion of quasi-analyticity from the tie with the given sequence $M=(M_n)_0^\infty$ of positive numbers.

1. The classes $C^M(\Omega)$

First, let us recall the following well known characterization of analytic functions.

LEMMA 1.1. *For any $f\in C^\infty(\Omega)$, the following two statements are equivalent:*

(a) $f\in A(\Omega)$.

(b) *For any compact set K in Ω, there are positive constants $C_0=C_0(K,f)$ and $C_1=C_1(K,f)$ such that*

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$$(1.1)\qquad \sup_K |f^{(n)}(x)| \le C_0 C_1^n n!,\quad n\ge 0.$$

Let $M=(M_n)_0^\infty$ be a sequence of positive numbers, which we assume only for simplicity (cf. [2, 3]) is such that

$$(1.2)\qquad M_0=1$$

$$(1.3)\qquad M_n^2 \le M_{n-1}M_{n+1},\quad n\ge 1.$$

DEFINITION 1.1. Let $C^M(\Omega)$ be the set of all functions f in $C^\infty(\Omega)$ such that for any compact set K in Ω, there are positive constant $C_0 = C_0(K,f)$ and $C_1=C_1(K,f)$ such that

$$(1.4)\qquad \sup_K |f^{(n)}(x)| \le C_0 C_1^n M_n,\quad n\ge 0.$$

The class $C^M(\Omega)$ is called to be quasi-analytic if it has the uniqueness property. If then, we call functions in $C^M(\Omega)$ to be quasi-analytic of class M.

When $M=(n!)$, we get just $C^M(\Omega)=A(\Omega)$. In W. Rudin[4], when $\Omega=R^1$, he requires the estimate (1.4) to hold globally on R^1, which is too much restrictive since if then, the function $f(x)=x$, for example, can not be in $C^M(R^1)$ for any M. For the simple proof of the following characterization of quasi-analytic classes, we refer to [2].

THEOREM 1.1. (Denjoy and Carleman [1]) *The following statements are all equivalent.*

(a) $C^M(\Omega)$ *is quasi-analytic.*

(b) $\sum_1^\infty M_n^{-1/n}=\infty$.

(c) $\sum_1^\infty M_{n-1}/M_n=\infty$.

Now, we shall give another criterion for $C^M(\Omega)$ to be quasi-analytic. To be precise, the class $C^M(\Omega)$ is quasi-analytic if and only if it has no non-trivial function with compact support. In fact, we shall prove the following which showes that there is a partition of unity subordinate to any open covering of Ω if $C^M(\Omega)$ is not quasi-analytic.

THEOREM 1.2. *The following statements are all equivalent.*

(a) $C^M(\Omega)$ *is not quasi-analytic.*

(b) *$C^M(\Omega)$ has a non-trivial function with compact support.*

(c) *For any compact set K in Ω, there is a function ϕ which has a compact support in Ω and is identically equal to 1 on K.*

It is trivial that the statements (b) or (c) implies (a) by the definition of quasi-analytic class. To show the other implications, let us note first that the quasi-analyticity depends only on the sequence $M=(M_n)$ but not on the domain Ω involved by the Denjoy-Carleman theorem.

LEMMA 1.2. *The class $C^M(\Omega)$ is closed under pointwise multiplication of functions.*

Proof. It follows immediately from the Leibnitz rule for differentiation and the logarithmic convexity of the sequence $M=(M_n)$, i.e., (1.3).

To see that (a) implies (b), we first assume that $\Omega=R^1$ and that $C^M(R^1)$ is not quasi-analytic. Then there is a function $f\in C^M(R^1)$, not identically 0, but $f^{(n)}(x_0)=0$, $n\geq 0$, at some point x_0 in R^1. Since $C^M(R^1)$ is invariant under any affine transformation, we may assume that $x_0=0$ and $f(z)=0$ for some $z>0$. Define $g(x)$ on R^1 by setting $g(x)=f(x)$ for $x\geq 0$ and $g(x)=0$ for $x<0$. Then $g\in C^M(R^1)$ has its support in $[0,\infty)$ and $g\neq 0$. Then the function $h(x)=g(x)g(2z-x)$ satisfies the requirements in (b). That is, (a) implies (b) when $\Omega=R^1$. For an arbitrary open set Ω in R^1, choose a function $f\in C^M(R^1)$, not identically 0, which has a compact support in R^1. We may assume by a suitable affine transformation that f has its support in Ω. Then f is in $C^M(\Omega)$. Finally, let us show that (b) implies (c). Assuming (b), we can choose a function f in $C^M(R^1)$, not identically 0, which has its support in $B=\{x\in R^1\,|\,|x|\leq 1\}$. Let us set $g(x)=f^2/\int_{R^1}f^2\,dx$. Then $g\in C^M(R^1)$ is such that $g(x)\geq 0$, $g\neq 0$, supp $g\subseteq B$ and $\int_{R^1}g(x)dx=1$. Let K be any compact subset of Ω and choose a number $\varepsilon>0$ so small that $|x-y|\geq 4\varepsilon$ for $x\in K$ and $y\notin\Omega$. Let $u(x)$ be a characteristic function of $K_{2\varepsilon}=\{y\in R^1\,||y-x|\leq 2\varepsilon$ for some $x\in K\}$. Let us define a function ϕ on R^1 by

$$\phi(x)=u*g_\varepsilon$$

where $g_\varepsilon(x)=\frac{1}{\varepsilon}g(x/\varepsilon)$.

Then $\phi \in C^\infty(\Omega)$ has its support in $K_{3\varepsilon}$ and is identically equal to 1 on K_ε. Lastly, we have $\phi \in C^M(\Omega)$ since for any x in Ω and any integer $n \geq 0$

$$|\phi^{(n)}(x)| = |u * g_\varepsilon^{(n)}| \leq \varepsilon^{-n} \int_B |g^{(n)}| dx \leq (C_0 \pi)(C_1 \varepsilon^{-1})^n M_n$$

where $C_0 = C_0(B, g)$ and $C_1 = C_1(B, g)$. It completes the proof of Theorem 1.2.

When the class $C^M(\Omega)$ is not quasi-analytic, the construction of partition of unity consisting of functions in $C^M(\Omega)$ subordinate to any open covering of Ω is now a straightforward generalization of the one in C^∞-category by the part (c) of Theorem 1.2. Thus for any non quasi-analytic class $C^M(\Omega)$, one can take the space of functions in $C^M(\Omega)$ with compact support as a space of test functions. With a suitable locally convex topology on it, C. Roumieu [3] developed the theory of generalized distributions, which is very much parallel to that of the usual distributions by L. Schwartz.

2. Quasi-analytic functions

Let us note that unlike quasi-analyticity, analyticity can be defined first for individual functions in Ω and then they form a subclass $A(\Omega)$ of $C^\infty(\Omega)$. It is a simple matter to free the notion of quasi-analyticity from the tie with the connection with the given sequence $M = (M_n)$ of positive numbers. We may simply call a subclass S of $C^\infty(\Omega)$ to be quasi-analytic if S has the uniqueness property. Let $Q(\Omega)$ be the union of all quasi-analytic subclasses of $C^\infty(\Omega)$. Then a function $f(x) \in C^\infty(\Omega)$ is in $Q(\Omega)$ if f is either identically equal to 0 or nowhere flat in Ω($f(x) \in C^\infty(\Omega)$ is flat at $x_0 \in \Omega$ if $f^{(n)}(x_0) = 0$, $n \geq 0$). Let us consider the function $f(x) = e^{-1/x} + 1$ for $x > 0$ and $= 1$ for $x \leq 0$. Then f is in $Q(\Omega)$ but $f(x) - 1$ and $f'(x)$ are not in $Q(\Omega)$. That is, the class $Q(\Omega)$ is invariant neither under the translation nor under the differentiation. In order to remedy such a unnaturalness let us note that the class $C^M(\Omega)$ is invariant under the perturbation of any polynomial since we may require the estimate (1.4) to hold for large n only without changing the class $C^M(\Omega)$ itself. Thus we have:

LEMMA 2.1. *The class $C^M(\Omega)$ is quasi-analytic if and only if any function $f(x) \in C^M(\Omega)$ is either a polynomial or $f(x) + p(x)$ is nowhere* *flat in Ω for any polynomial p.*

Proof. Necessity: If $f\in C^M(\Omega)$ is a polynomial, then there is nothing to prove. So, we assume that f is not a polynomial and $f+p$ is flat at some point x_0 in Ω. Then $f+p$ must be identically 0 in Ω since $C^M(\Omega)$ is quasi-analytic. It's a contradition. Sufficiency: If a function $f\in C^M(\Omega)$ is flat at some point x_0 in Ω, then it must be a polynomial and so be identically 0 in Ω.

It is trivial that the condition in Lemma 2.1 is equivalent to that for any $f(x)\in C^M(\Omega)$, we have either $f^{(n)}$ is identically 0 for some $n\geq 0$ or $f^{(n)}(x)$ is nowhere flat in Ω for all $n\geq 0$. It thus leads to the following definition:

DEFINITION 2.1. A function $f(x)\in C^\infty(\Omega)$ is quasi-analytic in Ω if either $f^{(n)}(x)$ is identically 0 for some $n\geq 0$ or $f^{(n)}(x)$ is nowhere flat in Ω for all $n\geq 0$. Let $QA(\Omega)$ be the set of all quasi-analytic functions in Ω.

By definition, the class $QA(\Omega)$ of quasi-analytic functions is now invariant under the differentiation and the perturbation by any polynomial. It's a proper subclass of $C^\infty(\Omega)$ which contains $A(\Omega)$.

For example, any function $f\in C^\infty(\Omega)$ such that the radius of convergence of its Taylor series at any point of Ω is 0 belongs to $\Omega A(\Omega)$ for if $f^{(n)}(x)$ is flat for some $n\geq 0$ at some point x_0 in Ω, then its Taylor expansion at x_0 has only finitely many terms, which is a contradiction.

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KAIST
P.O. Box 150, Cheongryang
Seoul 130-650, Korea