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A SUFFICIENT CONDITION FOR HYPOELLIPTICITY OF OPERATORS OF ORDER ONE

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1. Introduction

A linear differential operator P with C^∞ coefficients defined in an open set $\Omega \subset R^n$ is called to be hypoelliptic if for any $u \in D'(\Omega)$ and any open set $\Omega_1 \subset \Omega$, $Pu \in C^\infty(\Omega_1)$ implies that $u \in C^\infty(\Omega_1)$.

In [2], Radkevic and Oleinik gave a sufficient condition for the hypoellipticity of linear differential operators of any order satisfying a priori estimate. In this work, we shall give the same result assuming only the first 3 conditions of Radkevic and Oleinik when the differential operator is of order 1 (cf. Theorem 1. and Remark 1.). Our proof uses the method of microlocalized estimation (cf. [3]), which is considerably simpler than that of Radkevic and Oleinik.

We use the following notation for the symbol $p(x,\xi)$ of differential operator $P(x, D)$; for any multi-indices α and β,

$$p^{(\alpha)}(x,\xi)=\frac{\partial^{|\alpha|}p(x,\xi)}{\partial\xi^\alpha},\quad p_{(\alpha)}(x,\xi)=D_x^\alpha p(x,\xi),$$

$$p_{(\beta)}^{(\alpha)}(x,\xi)=\frac{\partial^{|\alpha|}D_x^\beta p(x,\xi)}{\partial\xi^\alpha},$$

$$p^{(j)}(x,\xi)=\frac{\partial}{\partial\xi_j}p(x,\xi),\quad p_{(j)}(x,\xi)=D_jp(x,\xi)\ \text{for}\ j=1,\cdots,n.$$

The operators $P^{(\alpha)}, P_{(\alpha)}, P_{(\beta)}^{(\alpha)}, P^{(j)}$ and $P_{(j)}$ are obtained from the corresponding symbols by replacing the vector ξ by the vector $(D_1,\cdots, D_n)$.

We denote by Ψ^m the space of all mth order pseudodifferential operators of classical type, $\Psi^{-\infty}=\bigcap_m\Psi^m$ and $\Psi^\infty=\bigcup_m\Psi^m$.

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2. Hypoellipticity

THEOREM 1. *Let Ω be an open subset in R^n. Let P be a first order differential operator on Ω with coefficients in $C^\infty(\Omega)$ for which the following conditions are fulfilled:*

I. *For each compact set $K\subset\Omega$ there exists a constant $s_0=s_0(K)$ such that for sufficiently large $N>0$ the inequality*

$$(2.1)\qquad \|u\|_{s_0}^2\leq c(K,N)\|u\|_{-N}^2+c(K)\|Pu\|_0^2$$

holds, where $-N<s_0$ and $u\in C_0^\infty(K)$.

II. *For each compact set $K\subset\Omega$, $s\in R^1$ and $\delta_1>0$, there exists a constant $c(K,s,\delta_1,N)$ such that for sufficiently large $N>0$ the inequality*

$$(2.2)\qquad \sum_{j=1}^n\|P_{(j)}u\|_s^2\leq\delta_1\|Pu\|_{s+1}^2+c(K,s,\delta_1,N)\|u\|_{-N}^2$$

holds, where $-N<s+s_0$, $u\in C_0^\infty(K)$.

III. *For each compact set $K\subset\Omega$, $s\in R^1$ and sufficiently large N, the inequality*

$$(2.3)\qquad \sum_{j=1}^n\|P^{(j)}u\|_s^2\leq c(K,s)\{\|Pu\|_{s-\mu}^2+c(N)\|u\|_{-N}^2\}$$

holds, where $\mu=\mu(K)>0$, $-N<s+s_0$, $u\in C_0^\infty(K)$.

Then for any $u\in D'(\Omega)$ such that Pu belongs to $H_{loc}^s(\Omega)$ we have the estimate

$$(2.4)\qquad \|\phi u\|_{s+s_0}^2\leq c(\phi_1)\{\|\phi_1 Pu\|_s^2+\|\phi_1 u\|_s^2\},$$

where the functions ϕ, $\phi_1\in C_0^\infty(\Omega)$, $\phi_1=1$ on supp ϕ. In particular, $P(x,D)$ is hypoelliptic.

REMARK 1. When $P(x,D)$ is of arbitrary order, it was shown in [2] that it is hypoelliptic if we assume, in addition to the three conditions in Theorem 1, that

IV. For each compact set $K\subset\Omega$, $\delta_1>0$, $s\in R^1$ and sufficiently large $N>0$, the inequality

$$(2.5)\qquad \sum_{j=1}^n\|P^{(j)}u\|_s^2\leq\delta_1\|Pu\|_s^2+C(\delta_1,N,s,K)\|u\|_{-N}^2$$

holds, where $-N<s+s_0$, $u\in C_0^\infty(K)$.

We need the following lemma which is proved in [2].

LEMMA 1. *If P satisfies II and III, then for each compact set $K\subset\Omega$, $s\in R^1$ and sufficiently large $N>0$ the inequality*

$$\|P_{(\beta)}{}^{(\alpha)}u\|_s^2\leq c(\alpha,\beta,s,K)\{\|Pu\|_{s+|\beta|-\mu}^2+c(N)\|u\|_{-N}^2\} \tag{2.6}$$

holds, where $|\alpha|\geq 1$, $u\in C_0^\infty(K)$.

Proof of Theorem 1. We first show that the estimate (2.1) can be localized, that is, $\|\phi u\|_{s_0}\leq c(\|\phi_1 Pu\|+\|\phi_1 u\|_{-N})$ for any $u\in C^\infty(\Omega)$, where ϕ and ϕ_1 are the same as in Theorem 1. By (2.1), we have

$$\begin{aligned}\|\phi u\|_{s_0}&\leq c(\|P\phi u\|+\|\phi u\|_{-N})\\ &\leq c(\|\phi Pu\|+\|[P,\phi]u\|+\|\phi u\|_{-N})\\ &\leq c(\|\phi_1 Pu\|+\|\phi_1 u\|_{-N}+\|[P,\phi]u\|).\end{aligned}$$

Choose $\{\psi_j\}_0^\infty\subset C_0^\infty(\Omega)$ such that $\psi_0=\phi$, supp $\psi_j\subset$supp $\psi_{j+1}\subset\cdots\subset$ supp ϕ_1 and $\psi_{j+1}=1$ on supp ψ_j.

Then

$$[P,\psi_j]u=\sum_{|\alpha|\geq 1}\frac{1}{\alpha!}D_x{}^\alpha\psi_j(x)P^{(\alpha)}u=\sum_{|\alpha|\geq 1}\frac{1}{\alpha!}D_x{}^\alpha\psi_j(x)P^{(\alpha)}\psi_{j+1}u.$$

We have, by Lemma 1, that

$$\begin{aligned}\|D_x{}^\alpha\phi(x)P^{(\alpha)}\psi_1 u\|\leq c\|P^{(\alpha)}\psi_1 u\|&\leq c(\|P\psi_1 u\|_{-\mu}+\|\psi_1 u\|_{-N})\\ &\leq c(\|\psi_1 Pu\|_{-\mu}+\|[P,\psi_1]u\|_{-\mu}+\|\psi_1 u\|_{-N})\\ &\leq c(\|\phi_1 Pu\|+\|\phi_1 u\|_{-N}+\|[P,\psi_1]u\|_{-\mu}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|D_x{}^\alpha\psi_1(x)P^{(\alpha)}\psi_2 u\|_{-\mu}&\leq c\|P^{(\alpha)}\psi_2 u\|_{-\mu}\\ &\leq c(\|\phi_1 Pu\|+\|\phi_1 u\|_{-N}+\|[P,\psi_2]u\|_{-2\mu}).\end{aligned}$$

Repeating the same process, we have

$$\begin{aligned}\|D_x{}^\alpha\psi_{l-1}(x)P^{(\alpha)}\psi_l u\|_{-(l-1)\mu}&\leq c\|P^{(\alpha)}\psi_l u\|_{-(l-1)\mu}\\ &\leq c(\|P\psi_l u\|_{-l\mu}+\|\psi_l u\|_{-N})\\ &\leq c(\|\psi_l u\|_{-l\mu+1}+\|\phi_1 u\|_{-N}).\end{aligned}$$

If l is so large that $-l\mu+1\leq -N$, then

$$\|\psi_l u\|_{-l\mu+1}\leq c\|\psi_l u\|_{-N}\leq c\|\phi_1 u\|_{-N}.$$

Consequently, we have

$$\|[P, \phi]u\| \leq c(\|\phi_1 Pu\| + \|\phi_1 u\|_{-N}).$$

Now we show that the estimate (2.1) can be microlocalized. Let $\varsigma(\delta)=\varsigma(\delta, x, D)=\psi(2\delta D)\phi(x)$ and $\varsigma_1(\delta)=\varsigma_1(\delta, x, D)=\psi(\delta D)\phi_1(x)$ where $\psi(\xi)\in C_0^\infty\left(|\xi|\leq\frac{3}{2}\right)$, $0\leq\psi(\xi)\leq 1$, $\psi(\xi)=1$ for $|\xi|\leq 1$. Then,

(2.7) $\varsigma_1(\delta, x, \xi)=1$ on supp $\varsigma(\delta, x, \xi)=\left\{(x, \xi)\,|\,x\in\text{supp}\,\phi,\ |\xi|\leq\frac{3}{4\delta}\right\}$.

(2.8) $\varsigma(\delta)$, $\varsigma_1(\delta)$ are bounded in Ψ^0 for $0<\delta<1$, self adjoint, and $\varsigma(\delta)$, $\varsigma_1(\delta)\in\Psi^{-\infty}$.

(2.9) $\varsigma(\delta)A\varsigma_1(\delta)=\varsigma(\delta)A+R_1(\delta)$ for any $A\in\Psi^\infty$, with $R_1(\delta)\in\Psi^{-\infty}$ bounded.

(2.10) $\varsigma_1(\delta)A\varsigma(\delta)=A\varsigma_1(\delta)+R_2(\delta)$ for any $A\in\Psi^\infty$, with $R_2(\delta)\in\Psi^{-\infty}$ bounded.

Of course, we assume that $\varsigma(\delta)$ and $\varsigma_1(\delta)$ are properly supported and supports of their distrubution kernels are contained in a sufficiently small neighborhood of the diagonal of $\Omega\times\Omega$. On the other hand, we can easily obtain that for any $u\in D'(\Omega)$, $\varsigma(\delta)u\in C_0^\infty(\Omega)$ and they have supports in a fixed compact subset of Ω independent of $\delta>0$.

Since $\{[P, \varsigma(\delta)]\,|\,0<\delta<1\}$ is bounded in Ψ^0,

$$\begin{aligned}(2.11)\qquad \|\varsigma(\delta)u\|_{s_0} &\leq c(\|P\varsigma(\delta)u\|+\|\varsigma(\delta)u\|_{-N})\\ &\leq c(\|\varsigma(\delta)Pu\|+\|[P, \varsigma(\delta)]u\|+\|\varsigma(\delta)u\|_{-N})\\ &\leq c(\|\varsigma_1(\delta)Pu\|+\|\varsigma_1(\delta)u\|+\|\varsigma_1(\delta)u\|_{-N})\\ &\leq c(\|\varsigma_1(\delta)Pu\|+\|\varsigma_1(\delta)u\|).\end{aligned}$$

Now we replace L^2-norms in (2.11) by H^s-norms. If $\Lambda^s=\text{op}(1+|\xi|^2)^{s/2}\in\Psi^s$, pseudodifferential operator with symbol $(1+|\xi|^2)^{s/2}$ which is modified to be properly supported, then we apply (2.1), with u replaced by $\Lambda^s u$, $\varsigma(\delta)$ by $\varsigma^s(\delta)=\Lambda^s\varsigma(\delta)\Lambda^{-s}$ and $\varsigma_1(\delta)$ by $\varsigma_1^s(\delta)=\Lambda^s\varsigma_1(\delta)\Lambda^{-s}$. Then we see that

$$\begin{aligned}(2.12)\qquad \|\varsigma(\delta)u\|_{s+s_0} &=\|\varsigma^s(\delta)\Lambda^s u\|_{s_0}\\ &\leq c(\|P\varsigma^s(\delta)\Lambda^s u\|+\|\varsigma^s(\delta)\Lambda^s u\|)\\ &\leq c(\|P\Lambda^s\varsigma(\delta)u\|+\|\varsigma(\delta)u\|_s)\\ &\leq c(\|[P, \Lambda^s\varsigma(\delta)]u\|+\|\Lambda^s\varsigma(\delta)Pu\|+\|\varsigma_1(\delta)u\|_s)\\ &\leq c(\|\varsigma_1(\delta)u\|_s+\|\varsigma_1(\delta)Pu\|_s).\end{aligned}$$

This completes the proof of the inequality (2. 4).

Now we will show that the inequality (2. 4) implies the hypoellipticity of $P(x, D)$. Assume that $Pu \in H_{loc}{}^{s}(\Omega)$. By schrinking Ω, we may assume that $u \in H_{loc}{}^{t}(\Omega)$ for some t. If $t \geq s$, then $u \in H_{loc}{}^{t}(\Omega)$ implies that $\|\varsigma(\delta)u\|_{s+s_0} < \infty$ independent of δ. Thus $u \in H_{loc}{}^{s+s_0}(\Omega)$.

If $t < s$, then $u \in H_{loc}{}^{t}(\Omega)$ and $Pu \in H_{loc}{}^{t}(\Omega)$ implies that $u \in H_{loc}{}^{t+s_0}(\Omega)$ by (2. 12). Continuing this process we have $u \in H_{loc}{}^{s+s_0}(\Omega)$. Since $C^{\infty}(\Omega) = \bigcap_{s \in R} H_{loc}{}^{s}(\Omega)$, $Pu \in C^{\infty}(\Omega)$ implies $u \in C^{\infty}(\Omega)$.

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