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ON THE AREA INTEGRAL OF HOLOMORPHIC L^p FUNCTIONS

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Let $D=\{z : |z|<1\}$ and $T=[0,2\pi]$. For f holomorphic in D and for $0<r\leq 1$, we let

$$A(r,f)=\iint_{|z|<r}|f'(z)|^2\, dxdy,\ \ z=x+iy,$$

and

$$A_r(t,\alpha,f)=\iint_{S(t,\alpha)\cap\{|z|<r\}}|f'(z)|^2\, dxdy,\ \ z=x+iy,$$

where $S(t,\alpha)$ is the interior of the convex hull of the circle $\{|z|=\sin\alpha\}$ and the point e^{it}. It is known that if $f\in H^p$ $(0<p\leq 2)$ then $A(r,f)=o(1-r)^{-2/p}$. See [1] and [2] for H^p spaces and the corresponding results. In the same vein is the following result of S. Yamashita:

THEOREM A. [4. Theorem 1.]. *Let f be a function holomorphic in D, and suppose that, for a p, $0<p<\infty$,*

$$\iint_D|f(z)|^p\, dxdy<\infty. \tag{1}$$

Then, for each $\alpha : 0<\alpha<\pi/2$,

$$\lim_{r\to 1}(1-r)^{2/p}A_r(t,\alpha,f)=0 \tag{2}$$

holds for a.e. $t\in T$.

The main idea of Yamashita's proof of Theorem A was the celebrated theorem of Marcinkiewicz and Zygmund [3. Theorem 1]. Our starting point here is the question whether the result of Theorem A is sharp. And, to find a proper converse of Theorem A is our goal in this note.

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Consider a slight weakening of (2), that is, suppose that for some α and $\gamma: 0<\alpha<\pi/2,\ 0<\gamma<2/p$, there is a constant M such that

$$\overline{\lim_{r\to 1}}(1-r)^{\gamma}A_r(t,\alpha,f)\leq M \quad \text{a.e. } t\in T.$$

Then, obviously,

$$\int_T dt\int_0^1(1-r)^{\delta-1}A_r(t,\alpha,f)dr<\infty \tag{3}$$

for all δ, $\gamma<\delta<2/p$.

We claim (1) under the condition (3). Then it will ensure that the exponent in (2) is best possible. Denote by $\chi_r(z)$ and $\chi_{S(t,\alpha)}(z)$ the characteristic functions of the sets $\{|z|<r\}$ and $S(t,\alpha)$ respectively. Note first that

$$\begin{aligned}&\int_0^1(1-r)^{\delta-1}A_r(t,\alpha,f)dr\\ &=\int_0^1(1-r)^{\delta-1}dr\iint_{S(t,\alpha)}\chi_r(z)|f'(z)|^2dxdy\\ &=\frac{1}{\delta}\iint_{S(t,\alpha)}(1-|z|)^{\delta}|f'(z)|^2dxdy,\end{aligned}$$

so that

$$\begin{aligned}&\int_T dt\int_0^1(1-r)^{\delta-1}A_r(t,\alpha,f)dr\\ &=\frac{1}{\delta}\int_T dt\iint_{S(t,\alpha)}(1-|z|)^{\delta}|f'(z)|^2dxdy.\end{aligned} \tag{4}$$

On the other hand, it is easy to see that

$$\int_T\chi_{S(t,\alpha)}(z)dt\sim 1-|z|$$

(Here, "$\sim$" means the equivalence of two quantities, that is, the quotient of the two quantities lies between two positive constants independent of z), so that the last integral of (4) is finite if and only if

$$\iint_D(1-|z|)^{\delta+1}|f'(z)|^2dxdy<\infty,$$

and this in turn is equivalent, by the Parseval's identity, to

(5) $$\iint_D (1-|z|)^{\delta-1}|f(z)|^2 dxdy < \infty.$$

Now, it is not difficult to see that (5) implies (1) if $0<p\leq 2$. The same argument gives (1) for $2<p<\infty$ if we consider (3) with $(1-r)^{p\delta/2-1} A_r(t,\alpha,f)^{p/2}$ in place of the integrand.

We state this as a theorem:

THEOREM 1. *Let f be holomorphic in D, and suppose that for some α and γ : $0<\alpha<\pi/2$, $0<\gamma<2/p$,*

(6) $$\operatorname*{ess\,sup}_{t\in T} \varlimsup_{r\to 1}(1-r)^{\gamma} A_r(t,\alpha,f)<\infty,$$

then

$$\iint_D |f(z)|^p dxdy < \infty.$$

COROLLARY. *The exponents of* (2) *in Theorem A and* (6) *of Theorem 1 are best possible.*

References

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