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A PROOF OF TREVES' ESTIMATES

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0. Introduction

Since the uniqueness in the Cauchy problem for partial differential equation was written by A.P. Calderon in 1958, Carleman's estimates have been used in the proof of the uniqueness in the Cauchy problem [2]. F. Treves proved the relation between uniqueness in the Cauchy problem and solvability of pseudo-differential equations using his estimates in [4].

We shall prove his estimates.

I. Preliminaries

We consider a differential operator $P(x, D)$ of order $m>0$, with C^∞ coefficients, in an open subset Ω of $\boldsymbol{R}^n$. We also consider a real valued C^∞ function ϕ in Ω. We shall use the notation $\phi'(x)=\text{grad }\phi(x)$. We introduce the pseudo-differential operator in $\Omega\times\boldsymbol{R}^1$,

$$(1) \qquad \tilde{P}_\phi=P(x, D_x+i\phi'(x)|D_t|).$$

It is defined by the formula:

$$(1.2) \qquad \tilde{P}_\phi u(x,t)=\frac{1}{2\pi}\int e^{i\sigma t}P(x, D_x+i\phi'(x)|\sigma|)\hat{u}(x,\sigma)d\sigma,$$

where $\hat{u}(x,\sigma)$ is the Fourier transform with respect to t of $u(x,t)$ $C_0^\infty(\Omega\times\boldsymbol{R}^1)$.

The following Lemma and Theorems are due to Treves [4].

LEMMA 1.1. *To every compact subset K of Ω there is a constant $C_K>0$ such that, for every real number τ, for every $f\in C_0^\infty(K)$ and every*

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$g\in C_0^\infty(\boldsymbol{R}^1)$, *we have:*

$$(1.3)\qquad \left| \|\tilde{P}_\phi(e^{i\tau t}f(x)g(t))\| - \|g\|\|P(x, D_x+i\phi'(x)|\tau|)f\| \right| \leq C_k \sum_{\alpha\neq 0} N_{|\alpha|}(g)\|P^{(\alpha)}(x, D_x+i\phi'(x)|\tau|)f\|$$

where

$$N_{|\alpha|}(g)=\left\{\frac{1}{2\pi}\sum_{k=1}^{|\alpha|}|\sigma|^{2k}|\hat{g}(\sigma)|^2 d\sigma\right\}^{1/2},\quad \alpha\in N^n.$$

In the statement below we shall use the following notation:

$$(1.4)\qquad \tilde{P}_\phi^{(\alpha)}=P^{(\alpha)}(x, D_x+i\phi'(x)|D_t|),\quad \alpha\in N^n$$

THEOREM 1.1. *Let x_0 be a point of Ω. Suppose that, for some constants C, $T>0$ the following is true:*

(1.5) *to every $\varepsilon>0$ there is an open neighborhood U_ε of x_0 in Ω such that, for every $u(x,t)\in C_0^\infty(U_\varepsilon\times]-T, T[)$,*

$$(1.6)\qquad \sum_{\alpha\neq 0}\varepsilon^{-|\alpha|}\|\tilde{P}_\phi^{(\alpha)}u\|\leq C\|\tilde{P}_\phi u\|$$

Then there is a constant $C'>0$ such that the following is true:

(1.7) *to every $\varepsilon>0$ there is an open neighborhood U_ε of x_0 in Ω such that, for every number $\tau>0$ and every $v(x)\in C_0^\infty(U_\varepsilon')$,*

$$(1.8)\qquad \sum_{\alpha\neq 0}\varepsilon^{-|\alpha|}\|e^{\tau\phi}P^{(\alpha)}(x, D_x)v\|\leq C'\|e^{\tau\phi}P(x, D_x)v\|.$$

THEOREM 1.2. *Let x_0 be a point of Ω. Suppose that, for some constant $C'>0$,* (1.8) *is true. Then there is a constant $C>0$ such that the following is true:*

(1.9) *to every $\varepsilon>0$ there is an open neighborhood U_ε of x_0 in Ω such that, for every $U(x,t)\in C_0^\infty(U_\varepsilon\times\boldsymbol{R}^1)$,* (1.6) *holds.*

We introduce a positive C^∞ function in $\boldsymbol{R}^1$, $h(\sigma)$, having the following properties:

$$(1.10)\qquad h(\sigma)\nearrow+\infty \text{ when } |\sigma|\nearrow+\infty$$

(1.11) there is a number $d\geq 0$ such that

$$(1+|\sigma-\tau|)^{-d}|h(\sigma)-h(\tau)|$$

is bounded in $\boldsymbol{R}^2$.

The simplest example is given by $h(\sigma)=(1+|\sigma|)^{-d},\ 0<d\leq 1$.

THEOREM 1.3. *Let x_0 be a point of Ω. Suppose that there are constants $C,\ T>0$ and an open neighborhood U of x_0 in Ω such that, for every $U(x,t)\in C_0^\infty(U\ \times]-T,T[)$, we have*

$$\sum_\alpha \|h(D_t)^{|\alpha|}\tilde{P}_\phi^{(\alpha)}u\|\leq C\|\tilde{P}_\phi u\| \tag{1.12}$$

Then there are constants $\tau_0, C'>0$ such that, for every $\tau>\tau_0$ and every $v(x)\in C_0^\infty(U)$.

$$\sum_\alpha h(\tau)^{|\alpha|}\|e^{\tau\phi}P^{(\alpha)}(x,D_x)v\|\leq C'\|e^{\tau\phi}P(x,D_x)v\| \tag{1.13}$$

II. Main theorems

We shall prove stronger estimates than (1.8) and (1.13).

THEOREM 2.1. *Let x_0 be a point of Ω. Suppose that, for some $T>0$, the following property holds:*

(2.1) *to every $\varepsilon>0$ there is an open neighborhood U_ε of x_0 in such that, for every $U\in C_0^\infty(U_\varepsilon\ \times]-T,T[)$,*

$$\sum_{|\alpha|+k\leq m-1}\|D_x^\alpha D_t^k u\|\leq\varepsilon\|\tilde{P}_\phi u\|. \tag{2.2}$$

Then, for some constant $C'>0$ the following is true:

(2.3) *to every $\varepsilon>0$ there is an open neighborhood U_ε' of x_0 in Ω such that, for every $\tau>1$ and every $v\in C_0^\infty(U_\varepsilon')$,*

$$\sum_{|\alpha|+k\leq m-1}\varepsilon^{|\alpha|+k-m}\ \tau^k\|e^{\tau\phi}D_x^\alpha v\|\leq C'\|e^{\tau\phi}P(x,D)v\|.$$

Proof. We take $U(x,t)=e^{i\tau t}f(x)g(t),\ f\in C_0^\infty(U_\varepsilon),\ g\in C_0^\infty(]-T,T[)$. We may assume for all $\varepsilon,\ 0<\varepsilon<1,\ U_\varepsilon$ is contained in a fixed compact subset K of Ω. By (1.3) and (2.2), we have

$$\begin{aligned}&\|g\|\|P(x,D_x+i\phi'(x)\tau)f\|\\ &\geq C_1\sum_{|\alpha|+k\leq m-1}\varepsilon^{|\alpha|+k-m}\|D_x^\alpha fD_t^t(e^{i\tau t}g(t))\end{aligned} \tag{2.4}$$

$$-C_1'\sum_{\alpha\neq 0}\sum_{k=1}^{|\alpha|}\|\tau^k\hat{g}(\tau)\|\|\|P^{(\alpha)}(x, D_x+i\phi'(x)\tau)f\|.$$

By Leibniz formula we obtain easily

$$\|D_t^k(e^{i\tau t}g(t))\|>\tau^k\|g\|. \tag{2.5}$$

Since $\tau^k\hat{g}(\tau)=(-1)^k\widehat{D_t^k g}(\tau)$, it follows from Plancherel's theorem that

$$\|\tau^k\hat{g}(\tau)\|\leq C\|g\| \text{ for some } C>0. \tag{2.6}$$

By (2.5) and (2.6) we have

$$\|P(x, D_x+i\phi'(x)\tau)f\|\geq C_1\sum_{|\alpha|+k\leq m-1}\varepsilon^{|\alpha|+k-m}\ \tau^k\|D_x^\alpha f\| \tag{2.7}$$
$$-C_2'\sum_{\alpha\neq 0}\|P^{(\alpha)}(x, Dx+i\phi'(x)\tau)f\|$$

Substituting

$$f=e^{\tau\phi}v$$

in (2.7) yields at once (2.3).

THEOREM 2.2. *Let the function h be as in Theorem* 1.3. *and let x_0 be a point of Ω. Suppose that there are constants C, $T>0$ and an open neighborhood U of x_0 in Ω such that, for every $U\in C_0^\infty(U\times]-T, T[)$, we have*

$$\sum_{|\alpha|+k\leq m-1}\|h(D_t)^{m-|\alpha|-k}D_x^\alpha D_t^k u\|\leq C\|\hat{P}_\phi u\| \tag{2.8}$$

Then there are constants $\tau_0, C'>0$ such that, for every $\tau>\tau_0$ and every $v\in C_0^\infty(U)$,

$$\sum_{|\alpha|+k\leq m-1}h(\tau)^{m-|\alpha|-k}\ \tau^k\|e^{\tau\phi}D_x^\alpha v\|\leq C'\|e^{\tau\phi}P(x, D)v\|. \tag{2.9}$$

Proof. We take $u(x, t)=e^{i\tau t}f(x)g(t)$, $f\in C_0^\infty(U)$, $g\in C_0^\infty(]-T, T[)$. Then we can easily compute

$$e^{-i\tau t}h(D_t)^j D_x^\alpha D_t^k(e^{i\tau t}f(x)g(t))=e^{-i\tau t}\tau^k h(D_t)^j\{e^{i\tau t}g(t)\} \tag{2.10}$$
$$D_x^\alpha f(x)+\sum_{l=1}^{k}\binom{k}{l}e^{-i\tau t}h(D_t)^j(e^{i\tau t}\tau^{k-l}D_g^l)D_x^\alpha f(x).$$

By proof of Theorem 1.2. in [4], we have

(2. 11) $\|e^{-i\tau t}h(D_t)^j\{e^{i\tau t}g(t)\}-h(\tau)^j g\|\leq B_j'\sum_{j=1}^{j}h(\tau)^{j-j'}\|g\|d_j'$

Where $\|\ \|_s$ is the norm in the s-th Sobolev space on the real line. If we apply (2. 10) with $D_g{}^l$ instead of g, then we have

(2. 12) $\|e^{-i\tau t}h(D_t)^j\{e^{i\tau t}D_g{}^l\}\|\leq B_j'\sum_{j'=0}^{j}h(\tau)^{j-j'}\|g\|_{l+dj'}$

By Lemma 1. 1., (2. 8), (2. 10) & (2. 12), we have

(2. 13) $$\sum_{j\geq 1}\|e^{i\tau t}\tau^k h(D_t)^j\{e^{i\tau t}g(t)\}D_x{}^\alpha f(x)\|\leq C\|g\|\|P(x,D_x+i\phi'(x)\tau)f\|+C'\|g\|\sum_{\alpha\neq 0}\|P^{(\alpha)}(x,D_x+i\phi'(x)\tau)f\|$$
$$+\sum_{j\geq 1}\sum_{l=1}^{k}\binom{k}{l}\tau^{k-1}B_j'\sum_{j'=0}^{j}h(\tau)^{j-j'}\|g\|_{l+d_{j'}}\|D_x{}^\alpha f(x)\|$$

Since $H^{2,m}(R^n)$ is imbedded in $L^2(R^n)$ ([1]&[3]), it follows from (2. 11) and (2. 13) that

(2. 14) $$\sum_{j\geq 1}h(\tau)^j\tau^k(1-B_g')\|D_x{}^\alpha f(x)\|$$
$$\leq C\|P(x,D_x+i\phi'(x)\tau)f\|+C'\sum_{\alpha\neq 0}\|P^{(\alpha)}(x,D_x+i\phi'(x)\tau)f\|$$
$$+\sum_{j\geq 1}(1+\tau)^k B_j'\sum_{j'=0}^{j}h(\tau)^{j-j'}\|D_x{}^\alpha f(x)\|$$

By the property of h, it follows from (2. 14) that for sufficiently large τ,

(2. 15) $$\sum_{j\geq 1}h(\tau)^j\tau^k\|D_x{}^\alpha f(x)\|\leq C'\|P(x,D_x+i\phi'(x)\tau)f\|+C'\sum_{\alpha\neq 0}\|P^{(\alpha)}(x,D_x+i\phi'(x)\tau)f\|$$

Since in the right hand side of (2. 15) the second sum does not exceed C_α times the first one. Substituting $f=e^{\tau\phi}v$ and $j=m-|\alpha|-k$ in (2. 15) yields at once (2. 9).

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