Comm. Korean Math. Soc. 3(1988), No. 2, pp. 143~146

MODIFICATIONS OF ZF EQUICONSISTENT WITH ZF^{KM}

S.M. KIM

In a paper[3] W. Marek and A. Mostowski set forth an interesting extension of ZF (Zermelo-Fraenkel set theory), denoted as ZF^{KM} : ZF^{KM} is the set of formulas Φ of the language of ZF set theory such that the relativisation Φ^V of Φ to the universe of sets is provable in KM (Kelley-Morse theory, i.e., the impredicative extension of von Neumann-Bernays-Gödel set-theory (VBG)). Then the system ZF^{KM} is axiomatisable, but no axiomatisation (in the language of set theory) is known. They conjectured that ZF^{KM} consists of sentences as true as those of ZF set theory. This reminds us of the fact that ZF is equivalent to VBG and hence ZF is equiconsistent with VBG [6][7] and [4][5], respectively. Now it is natural to ask how ZF^{KM} proof-theoretically and/or model-theoretically compares with ZF or some other modifications of ZF. First however, we comment on notations. For the most part, our notation will be that commonly employed in set theory. In general, $\alpha, \beta, \gamma\cdots$ denote ordinals. $\lim(\alpha)$ stands for "α is a limit ordinal." $\mathrm{fn}(f)$ means that f is a function. Denote $\mathrm{dom}(f)$ for domain of f. If a is a set, $\mathrm{p}(a)$ denotes its power set. con(ZF) stands for "ZF is consistent." We refer the reader to [2] for any notion we do not cover. We present with the proof assumed:

LEMMA 1.

$$con(ZF^{KM}) \leftrightarrow con(KM)$$

Let us denote the second order ZF set theory by ZF_{II}. If we define ZF'_{II} to be the ZF_{II} plus the formula $(\forall A\ \forall B(\forall a(A(a) \equiv B(a)) \rightarrow A=$

Received August 15, 1988.
Presented at the Knoxville, Tennessee Meeting of the American Mathematical Society, March 25, 1988 (#841-04-74).
I would like to acknowledge KOSEF research grant.

B)) then we obtain:

THEOREM 2.

$$con(KM) \leftrightarrow con(ZF'_{\mathrm{II}})$$

Proof.

(1) Proof theoretical equivalence:

$$\begin{aligned}
&(q(a, \ldots) \wedge \forall b((q(b, \ldots) \to q(b, \ldots)) \wedge (\neg q(b, \ldots) \to \neg\, q(b, \ldots)))) \\
&\quad \to \exists p(pa \wedge \forall b((q(b, \ldots) \to pb) \wedge (\neg q(b, \ldots) \to \neg pb))) \\
&\quad \to (\mathrm{ZF}_{\mathrm{II}} \vdash q(a, \ldots) \to \exists p \forall b(q(b, a_1, \cdots, a_k, p_1, \cdots, p_j) \equiv pb)) \wedge \\
&\qquad (\mathrm{ZF}_{\mathrm{II}} \vdash \neg q(a, \ldots) \to \exists p \forall b(q(b, a_1, \cdots, a_k, p_1, \cdots, p_j) \equiv pb)) \\
&\quad \to (\mathrm{ZF}_{\mathrm{II}} \vdash \forall a_1 \forall a_2 \cdots \forall a_k \forall p_1 \forall p_2 \cdots \forall p_j \exists p(q(b, a_1, \cdots, a_k, p_1, \cdots, p_j) \equiv pb)),
\end{aligned}$$

where $q(a, a_1, \cdots, a_k, p_1, \cdots, p_j)$ is a formula in the language of $\mathrm{ZF}_{\mathrm{II}}$. The next step is to derive the following argument: if a formula $p(a, \cdots)$ from KM satisfies the substitution restriction for A then $\mathrm{KM} \vdash \forall A$ $q(A) \to q(p(a, \cdots))$), where $q(A)$ is a formula from KM and A free. This follows with the aid of the equivalence theorem of quantification. The above arguments suffice to put the comprehension schema and other axioms of KM in relation to the substitution schema and other axioms of $\mathrm{ZF}'_{\mathrm{II}}$, respectively.

(2) Equiconsistency:

If $\mathrm{ZF}'_{\mathrm{II}}$ were not consistent, two contradictory formulas, p and $\neg p$, could be derived from its axioms. Since contradictory formulas imply all wellformed formulas, we have that all formulas of $\mathrm{ZF}'_{\mathrm{II}}$ are deducible from the axioms of $\mathrm{ZF}'_{\mathrm{II}}$. By (1), all formulas of KM would be deducible from the axioms of KM. The converse follows in a similar manner.

Now let us consider the two modifications of ZF set theory, denoted $\underline{\mathrm{ZF}}$ and $\overline{\mathrm{ZF}}$, as follows.

DEFINITION 3.

(1) Let $\underline{\mathrm{ZF}}$ be the theory based on the following sentences:

1. $(\forall a)(\forall x)(\forall y)[x = y \wedge x \in a \to y \in a]$
2. $(\forall a)(\forall b)\mathscr{M}(\{a, b\})$.
3. $(\forall a)\mathscr{M}(\bigcup(a))$.
4. $(\forall a)[(\forall u)(\forall v)(\forall w)]\varphi(u, v) \wedge \varphi(u, w) \to v = w]$
   $\to (\exists b)(\forall y)[y \in b \leftrightarrow (\exists x)[x \in a \wedge \varphi(x, y)]]]$.

5. $(\forall a)[a\neq 0\rightarrow(\exists x)[x\in a\wedge x\cap a=0]]$.
6. $\mathscr{M}(\omega)$.
7. $(\exists f)(\exists\alpha)[\lim(\alpha)\wedge\alpha>\omega\wedge\mathrm{fn}(f)\wedge\mathrm{dom}(f)=\alpha+1$
   $\wedge f(0)=0\wedge(\forall\beta)[\beta<\alpha\rightarrow(\forall u)(u\subset f(\beta+1)\leftrightarrow u\subset f(\beta))]$
   $\wedge(\forall\lambda)[\lim(\lambda)\wedge\lambda<\alpha+1\rightarrow f(\lambda)=\bigcup_{\beta<\lambda}f(\beta)]$
   $\wedge(\forall x)(\forall g)(x\in f(\alpha)\wedge g\subset f(\alpha)\wedge\mathrm{fn}(g)\rightarrow g(x)\in f(\alpha)))$

Note that $\mathscr{M}(A)$ stands for "A is a set."

(2) Let $\overline{\mathrm{ZF}}$ be the theory based on the following sentences:

The first six are identical to the previous sentences 1 through 6 of $\underline{\mathrm{ZF}}$.

7. $\exists$ inaccessible cardinal.
8. $\forall a(a$: set constructed by an ordinal less than the first inaccessible cardinal$\rightarrow\mathscr{M}(p(a)))$.

Then it turns out that the two systems $\underline{\mathrm{ZF}}+V=L$ and $\overline{\mathrm{ZF}}+V=L$ are proof-theoretically equivalent. As a consequence of this equivalence we have:

THEOREM 4.

$$con(\underline{ZF}+V=L)\leftrightarrow con(\overline{ZF}+V=L)$$

Proof.

The proof is similar to the second part of that of Theorem 2.

Moreover, we obtain:

THEOREM 5.

$$con(\overline{ZF}+V=L)\leftrightarrow con(KM)$$

Proof.

In view of the definition of $\overline{\mathrm{ZF}}+V=L$, we see that

$$(\overline{\mathrm{ZF}}+V=L)\vdash\mathrm{con}(\mathrm{KM}).$$

On the other hand, con(KM) implies con(VBG+AC).

$$\mathrm{VBG}+\mathrm{AC}\vdash\mathrm{con}(\overline{\mathrm{ZF}}+V=L).$$

Thus we have established:

COROLLARY 6.

$$con(ZF^{KM}) \leftrightarrow con(KM) \leftrightarrow con(ZF'_{II}) \leftrightarrow con(\underline{ZF} + V=L) \leftrightarrow con(\overline{\overline{ZF}} + V=L)$$

Proof.
Lemma 1, Theorems 2, 4 and 5.

The results of this paper might give new insight into dealing with several open problems, e.g., the conservative extension problem [1, p. 246, Question 7.4(i)], and finding the axiomatisation of the ZF^{KM} set theory and some meaningful statements provable in ZF^{KM} but not in ZF; but these would require further research.

References

1. T.B. Flannagan, *Axioms of choice in Morse-Kelley class theory*, Springer-Verlag Lecture Note in Mathematics **499**, 1975, 190–247.
2. T. Jech, *Set Theory*, Academic Press, New York, 1978.
3. W. Marek and A. Mostowski, *On extendability of models of ZF set theory to the models of Kelley-Morse theory of classes*, Springer-Verlag Lecture Note in Mathematics **499**, 1975, 460–542.
4. A. Mostowski, *Some impredicative definitions in axiomatic set theory*, Fundamenta Mathematicae, 1950, 111–124.
5. I. Novak, *A construction for models of consistent systems*, Fundamenta Mathematicae, 1950, 87–110.
6. J.R. Schoenfield, *A relative consistency proof*, The Journal of Symbolic Logic, 1954, **19**, 21–28.
7. H. Wang, *On Zermelo's and Von Neumann's axioms for set theory*, Proceedings of the National Academy of Sciences USA, Vol. **35**, 1949, 150–155.

Yonsei University
Seoul 121–749, Korea