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*A NOTE ON THE CLIFFORD MODULES

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1. Introduction

It is well known that the theory of Clifford moudles is used to prove the Bott periodicity theorems in K-theory. There are difficult parts to understand in the theory of Clifford modules, for example Corollary 3.3. The purpose of this note is to introduce Theorem 3.2 and to prove Corollary 3.3 using Proposition 2.1 and Theorem 3.2.

2. Basic concepts and notations

Let $\boldsymbol{R}^n$ be the n-dimensional real space $(n\geq 0)$. For $x, y\in\boldsymbol{R}^n$ let $\langle x, y\rangle$ be the inner product defined by $x_1y_1+\cdots+x_ny_n$, where $x=(x_1, \cdots, x_n)$ and $x=(y_1, \cdots y_n)$. Then the *quadratic form* $(\boldsymbol{R}^n, Q)$ is defined by a function $Q:\boldsymbol{R}^n\to\boldsymbol{R}$ satisfying

(i) for $a\in\boldsymbol{R}$, $x\in\boldsymbol{R}^n$ $Q(ax)=a^2Q(x)$

(ii) for $x\in\boldsymbol{R}^n$

$$Q(x)=-\langle x, x\rangle$$

is a symmetric bilinear form ([2], [3], [4]).

The *Clifford algebra* of the quadratic from $(\boldsymbol{R}^n, Q)$ is a pair $(C(Q), \theta)$, where $C(Q)$ is a R-algebra and $\theta:\boldsymbol{R}^n\to C(Q)$ is a linear function such that for all $x\in\boldsymbol{R}^n$ $(\theta(x))^2=Q(x)\cdot 1$ where 1 is the unit element of $C(Q)$. Moreover, $(C(Q), \theta)$ satisfies the following universal property. For any R-algebra A and any homomorphism on the underlying vector spaces $\varphi:\boldsymbol{R}^n\to A$ such that $(\varphi(x)^2=Q(x)\cdot 1$, where $x\in\boldsymbol{R}^n$ and 1 is the unit element of A, there is a unique algebra homomorphism $\psi:C(Q)\to A$ making the following diagram commutative

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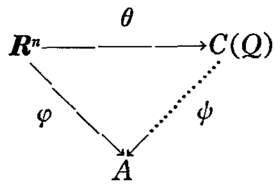

In the sequel, we shall put $(C(Q), \theta)=C_n$. If we take

(ii) $$Q(x)=\langle x, x\rangle \text{ for } x, y\in R^n$$

instead of (ii) above, we shall put $(C(Q), \theta)=C_n'$.

In $\boldsymbol{R}^n$, we put

$$e=(1, 0, \cdots, 0), \cdots\cdots\cdots\cdots, e_n=(0, \cdots, 0, 1)$$

Then $\{e_1, \cdots, e_n\}$ generates $\boldsymbol{R}^n$. Consider the tensor algebra $T(\boldsymbol{R}^n)=\sum_{k>0} T^k(\boldsymbol{R})=\sum_{k>0} a_{i_1\cdots i_k}\, e_{i_1}\otimes\cdots\otimes e_{i_k}$, where $a_{i_1\cdots i_k}\in\boldsymbol{R}$ and $1\leq i_1, \cdots, i_k\leq n$, and the two-side ideal $I(Q)$ generated by the elements $x\otimes x-Q(x)\cdot 1$ in $T(\boldsymbol{R}^n)$. Then

$$C_n=T(\boldsymbol{R}^n)/I(Q)$$

Thus there is the canonical epimorphism

$$\eta : T(\boldsymbol{R}^n)\longrightarrow C_n$$

If we put $\eta(\sum_{k>0} T^{2k}(\boldsymbol{R}^n))=C_n^0$ and $\eta(\sum_{k\geq 0} T^{2k+1}(\boldsymbol{R}^n))=C_n^1$ it is clear that

$$C_n=C_n^0\oplus C_n^1,$$

which is a Z_2-grading, where

$$T^{2k}(\boldsymbol{R}^n)=\sum a_{i_1\cdots i_{2k}}\, e_{i_1}\otimes\cdots\otimes e_{i_{2k}}\subset T(\boldsymbol{R}^n)$$

$1\leq i_1, \cdots, i_{2k}\leq n$ and $a_{i_1\cdots i_{2k}}\in\boldsymbol{R}$. Therefore

$$\dim_R C_n=2^n$$

and 1 together with the products $e_{i_1}\otimes\cdots\otimes e_{i_k}=e_{i_1}\cdots e_{i_k}$ form a base of C_n. We have to note that

$$e_i^2=-1 \text{ and } e_ie_j+e_je_i=0$$

in C_k, where $1 \leq i \neq j \leq k$.

A $\boldsymbol{Z}_2$-graded module M over the $\boldsymbol{Z}_2$-graded algebra C_k is a C_k-module M with $M = M^0 \oplus M^1$ such that for $i, j \in \boldsymbol{Z}_2$

$$C_k{}^i M^j \subset M^{i+j}.$$

A *Clifford module* is a $\boldsymbol{Z}_2$-graded module over a clifford algebra.

Let $\mathscr{M}(C_k)$ $(=\mathscr{M}_k)$ denote the free abelian group with irreducible $\boldsymbol{Z}_2$-graded C_k-module as free generators, and let $\mathscr{N}(C_k)$ denote the free abelian group with irreducible C_k as free generators.

We have the following isomorphisms ([1], [2]) :

(i) For $x \in \boldsymbol{R}^k - \{0\}$, $A(x) : C_k \xrightarrow{\cong} C_k$ is the inner automorphism $A(x)(a) = xax^{-1}$ and $A(x) | C_k{}^0 = A(x)^0$, where x^{-1} is the inverse of x which is defined as follows.

For each $x = a_1e_1 + \cdots + a_ke_k$

$$(a_1e_1 + \cdots + a_ke_k) \cdot \left(\frac{a_1e_1 + \cdots + a_ke_k}{a_1{}^2 + \cdots + a_k{}^2} \right) = -1$$

and thus

$$x^{-1} = (a_1e_1 + \cdots + a_ke_k)^{-1} = -\frac{a_1e_1 + \cdots + a_ke_k}{a_1{}^2 + \cdots + a_k{}^2}.$$

(ii) $\beta : C_k{}^0 \oplus C_k{}^0 = C_k \xrightarrow{\cong} C_k{}^0 \oplus C_k{}^1 = C_k$,

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \text{by } x_0 + x_1 & \longmapsto & x_0 - x_1 \end{array}$$

(iii) $\phi : C_k \longrightarrow C_{k+1}^0$ which is defined by $\phi(x_0 + x_1) = x_0 + e_{k+1}x_1$ for each $x_0 \in C_k{}^0$ and $x_1 \in C_k{}^1$.

(iv) For each C_k-module $M = M^0 \oplus M^1$ where $C(M)^0 = M^1$ and $C(M)^1 = M^1$.

For each graded C_k-module $M = M^0 \oplus M^1$ we define a functor $\boldsymbol{R}$ such that $\boldsymbol{R}(M) = M^0$, where M^0 is a $C_k{}^0$-module. Then $R : \mathscr{M}_k \to \mathscr{N}(C_k{}^0)$ is a group-homomorphism for each $k \geq 0$ ([2]).

PROPOSITION 2. 1. *The following diagram are commutative* :

$$\begin{array}{ccccc} \mathscr{M}_{k+1} & \xleftarrow{\;1\;} & \mathscr{M}_{k+1} & \xrightarrow{\;R\;} & (C_{k+1}^0) \\ \downarrow c & & \downarrow A(x)^* & & \downarrow A(x)^{0*} \\ \mathscr{M}_{k+1} & \xleftarrow{\;1\;} & \mathscr{M}_{k+1} & \xrightarrow{\;R\;} & (C_{k+1}^0) \end{array}$$

for $x \in R^k - \{0\}$.

Poof. Let M be a generator of $\mathcal{M}_{k+1}$. In $A(x)*M=Z$, we have the following: for each $\alpha \in C_k$

$$\pm\alpha Z \begin{cases} \text{Example; Let } x=e_i,\ \alpha\in C_k^0 \\ \text{If } e_i \text{ is not a factor of } \alpha, \text{ then } +\alpha Z. \\ \text{If } e_i \text{ is a factor of } \alpha, \text{ then } -\alpha Z. \end{cases}$$

because $\alpha Z = x\alpha x^{-1}z$. If we put $Z=xM$, then

$$\pm\alpha(xM) \begin{cases} \text{Example; Let } x=e_i,\ \alpha\in C_k^0 \\ \text{If } e_i \text{ is not a factor of } \alpha, \text{ then } +\alpha(xM). \\ \text{If } e_i \text{ is a factor of } \alpha, \text{ then } -\alpha(xM). \end{cases}$$

Therefore $A(x)*M=Z=xM$. Thus

$$\begin{aligned} Z^0 &= xM^0 = M^1 \text{ and } z^1 = xM^1 = M^0 \\ &= C(M)^0 \qquad\qquad\quad = C(M)^1 \end{aligned}$$

This implies that the left square in the above diagram is commutative.

For each generator $M=M^0 \oplus M^1$ of $\mathcal{M}_{k+1}$

$$A(x)*R(M)=M^1 \text{ and } R\ A(x)*(M)=R(M^1\oplus M^0)=M^1$$

and thus the right square is also commutative.

3. Main results

The tensor product of two $\boldsymbol{Z}_2$-graded algebras C_k and C_l, denoted $C_k \hat{\otimes} C_l$, is the tensor product of the underlying modules with

$$(C_k\hat{\otimes}C_l)^0 = C_k^0 \otimes_R C_l^0 \oplus C_k^1 \otimes_R C_l^1,$$
$$(C_k\hat{\otimes}C_l)^1 = C_k^0 \otimes_R C_l^1 \oplus C_k^1 \otimes_R C_l^0$$

and the multiplication given by

$$(x\hat{\otimes}y)(x'\hat{\otimes}y') = (-1)^{ij}(xx')\hat{\otimes}(yy')$$

for $x' \in C_k^i$ and $y \in C_l^i$. Note that for $x \in C_k^j$ we have $xy=(-1)^{ij}\,yx$.

An isomorphism

$$\varphi_{k,l} : C_{k+l} \longrightarrow C_k \hat{\otimes} C_l$$

is defined by the relations

$$\varphi_{k,l}(e_i)=e_i\hat{\otimes}1 \text{ for } 1\leq i\leq k$$

and

$$\varphi_{k,l}(e_j)=1\hat{\otimes}e_{j-k} \text{ for } k\leq j\leq k+1$$

DEFINITION 3. 1. We define an antiautomorphism

$$T: C_k\hat{\otimes}C_l\longrightarrow C_l\hat{\otimes}C_k$$

and an automorphism

$$\sigma: C_{k+l}\longrightarrow C_{k+l}$$

as follows. For $x_i\hat{\otimes}y_j\in C_k{}^i\hat{\otimes}C_l{}^j$, $T(x_i\hat{\otimes}y_j)=(-1)^{ij}y_j\otimes x_i$, and $\sigma=\varphi_{l,k}{}^{-1}\cdot T\cdot\varphi_{k,l}$ where $\psi_{k,l}$ and $\psi_{l,k}$ are given in the following diagram:

$$\begin{array}{ccc} C_{k+l} & \xrightarrow{\varphi_{k,l}} & C_k\hat{\otimes}C_l \\ & \varphi_{l,k}\searrow \quad \swarrow T & \\ & C_l\otimes C_k & \end{array}$$

THEOREM 3. 2. *For $e_i, e_j, e_t\in C_{k+l}$ $(1\leq i, j, t\leq k+l)$ and $y=(e_i+e_j)/\sqrt{2}$ the following holds.*

(i) *If $j=i+1$, then $A(y)e_i=e_{i+1}$, $A(y)e_{i+1}=e_i$*

(ii) *If $i\neq j$ and $i\neq t\neq j$, then $A(y)e_t=-e_t$.*

(iii) *If $i\neq j$ then $A(y)e_i=e_j$ and $A(y)e_j=e_i$.*

Proof. It is clear that for $i\neq j$

$$\left(\frac{e_i+e_j}{\sqrt{2}}\right)^{-1}=-\frac{e_i+e_j}{\sqrt{2}}$$

(i) $$A(y)e_i=\frac{e_i+e_i+1}{\sqrt{2}}(e_i)\left(-\frac{e_i+e_{i+1}}{\sqrt{2}}\right)=\frac{(e_i+e_{i+1})(1-e_ie_{i+1})}{2}$$

$$=\frac{e_i+e_{i+1}+e_{i+1}-e_i}{2}=e_{i+1},$$

and similarly,

$$A(y)e_{i+1}=e_i.$$

(ii)
$$A(y)e_t=\frac{e_i+e_j}{\sqrt{2}}(e_t)\left(-\frac{e_i+e_j}{\sqrt{2}}\right)$$
$$=\frac{e_i+e_j}{\sqrt{2}}\cdot\left(-\frac{e_i+e_j}{\sqrt{2}}\right)(-e_t)=-e_t.$$

(iii) By the same way as in proof (i) we can easily obtain.

$$A(y)e_i=e_j \text{ and } A(y)e_j=e_i.$$

Let M be a $\boldsymbol{Z}_2$-graded C_k-module and let N be a $\boldsymbol{Z}_2$-graded C_l-module. The $\boldsymbol{Z}_2$-graded tensor product $M\widehat{\otimes}N$ is the ordinary tensor product with the grading

$$(M\widehat{\otimes}N)^0=M^0\otimes_R N^0\oplus M^1\otimes_R N^1$$

and

$$(M\widehat{\otimes}N)^1=M^0\otimes_R N^1\oplus M^1\otimes_R N^0$$

and with the following multiplication by $C_k\widehat{\otimes}C_l$:

$$(a\widehat{\otimes}b)(x\widehat{\otimes}y)=(-1)^{ij}ax\widehat{\otimes}by \text{ for } b\in C_l^i \text{ and } x\in M^j.$$

Take generators $M\in\mathscr{M}_k$ and $N\in\mathscr{M}_l$, and recall the isomorphism

$$\varphi_{k,l} : C_{k+l} \cong C_k\widehat{\otimes}C_l.$$

Then

$$\varphi_{k,l}{}^* : \mathscr{M}_k\otimes_Z\mathscr{M}_l\longrightarrow\mathscr{M}_{k+l}$$

is a group homomorphism and thus

$$\mathscr{M}_*=\bigoplus_{k\geq 0}\mathscr{M}_k$$

becomes a grading ring ([1], [2]).

COROLLARY 3.3. *For* $u\in\mathscr{M}_k$ *and* $v\in\mathscr{M}_l$

$$uv=\begin{cases} vu & \text{if } kl \text{ is even}\\ c(vu) & \text{if } kl \text{ is odd.}\end{cases}$$

Proof. At first we have to note that

(i) $C=A(y)^*$ $(y\in\boldsymbol{R}^k-0)$

(ii) $C^k=\begin{cases}1 & \text{if } k \text{ is even}\\ C & \text{if } k \text{ is odd}\end{cases}$

by Proposition 2. 1.

By Definition 3. 1

$$\sigma(e_i)=\varphi_{l,k}{}^{-1}\circ T\circ\varphi_{k,l}(e_i)=\begin{cases}\varphi_{l,k}{}^{-1}\circ T(e_i\hat{\otimes}1)=e_{i+l} \text{ if } 1\leq i\leq k\\ \varphi_{l,k}{}^{-1}\circ T(1\hat{\otimes}e_{i-k})=e_{i-k} \text{ if } k\leq i\leq k+l\end{cases}$$

Our proof is divided into two steps.

Step I. The case when kl is even:

(i) k and l are both even.

By (i) of Theorem 3. 2

$$\sigma(e_i)=A(y_l)\cdots A(y_1)e_i=e_{i+l} \qquad (1\leq i\leq k)$$

and

$$\sigma(e_i)=A(y_k')\cdots A(y_1')e_i=e_{i-k} \qquad (k\leq i\leq k+l)$$

where

$$y_1=\frac{e_i+e_{i+1}}{\sqrt{2}}\cdots,\quad y=\frac{e_{i+l-1}+e_{i+l}}{\sqrt{2}}$$

and

$$y'=\frac{e_{i-1}+e_i}{\sqrt{2}},\ \cdots,\ y_k'=\frac{e_{i-k}+e_{i-k+1}}{\sqrt{2}}$$

That is, $\sigma^*=C^l$ or $\sigma^*=C^k$ and thus $\sigma^*\equiv C^{ke}$ in this case.

(ii) k is even and l is odd.

For $e_i(1\leq i\leq k)$

$$A(y_1)e_i=e_{i+1} \text{ where } y_1=e_i+e_{i+1}/\sqrt{2}$$

and we take the action of (ii) of Theorem 3. 2 of k-times. Repeatting this way, we have $\sigma^*=C^{kl}$. For $e_j(k\leq j\leq k+l)$ $\sigma^*=C^k=C^{kl}$.

For $\sigma(e_ie_j)$ $(1\leq i\leq k,\ k\leq j\leq k+l)$ we have

$$\sigma(e_ie_j)=\sigma(e_i)\sigma(e_j)$$

and thus

$$\sigma^*=C^{kl}\cdot C^k=C^{kl} \ (k : \text{even}).$$

For other cases, it is clear that $\sigma^*=C^{kl}$.

If k is odd and l is even, then by the same method as above we have $\sigma^*=C^{kl}$. Since $\sigma=\varphi_{lk}^{-1}\cdot T\cdot\varphi_{k,l}$ $\varphi_{l,k}\cdot\sigma=T\cdot\varphi_{kl}$ and σ^* $\varphi_{l,k}{}^*=\varphi_{k,l}{}^*\circ T^*$. Let M be a generator of $\mathscr{M}_k$ and let N be a generator of $\mathscr{M}_l$.
Then

$$\sigma^*\varphi_{l,k}{}^*(M\widehat{\otimes}N)=\varphi_{k,l}{}^*T^*(M\widehat{\otimes}N)$$

and thus

$$C^{kl}\varphi_{l,k}{}^*(M\widehat{\otimes}N)=\varphi_{k,l}{}^*(N\widehat{\otimes}M).$$

Since kl is even and $C^{kl}=1$ we have

$$\varphi_{l,k}{}^*(M\widehat{\otimes}N)=\varphi_{k,l}{}^*(N\widehat{\otimes}M).$$

Step II. The case when kl is odd, i.e., k and l are both odd. Note that $kl\equiv k\equiv l$ (mod. 2).

For $e_i(1\leq i\leq k)$ and $e_j(k\leq j\leq k+l)$ $\sigma(e_i)$ is equal to inner automorphism of l-times and $\sigma(e_j)$ is the same as inner automorphisms of k-times. Hence in this case, $\sigma^*=C^k=C^l=C^{kl}=C$.

For $\sigma(e_ie_j)$, by (i) and (ii) of Theorem 3.2 we have the following:

$$\sigma(e_ie_j)=\sigma(e_i)\sigma(e_j)=\begin{cases}\text{inner automorphisms of } l(k-1)+l=kl\text{-times} \\ \quad \text{if } 1\leq i\neq j\leq k \\ \text{inner automorphisms of } l(k-1)+k \text{ times} \\ \quad \text{if } 1\leq i\leq k \text{ and } k\leq j\leq k+l\end{cases}$$

In this case, since $l(k-1)+k\equiv kl$ (mod. 2) we have

$$\sigma^*=C^{kl}=C.$$

For other case, we use the same method as above.
Consequently, $\sigma^*=C$ when k and l are both odd.

In general, we have $\sigma^*=C^{kl}$ and our proof is complete.

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