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PRIME NEAR-RINGS

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1. Introduction

R.E. Johnson [2] obtained many interesting results in the theory of prime modules and annihilator ideals.

In this paper, we consider the basic properties of prime near-ring modules and find a similarity of prime submodules and annihilator left ideals. The main result is as follows: If N is a distributive near-ring, then the lattices $\mathscr{U}_r$ and $\mathscr{U}_l$ are anti-isomorphic.

2. Preliminaries on near-rings

We begin with some definitions and some results.

Throughout this paper, N will denote a (right) near-ring. A *(near-ring) module* is an algebraic system ${}_NM$ which is an additive group and N is a near-ring, together with a mapping $(n, m)\mapsto nm$ from $N\times M$ into M with the following properties:

$$(n_1+n_2)m=n_1m+n_2m \quad \text{and}$$
$$(n_1n_2)m=n_1(n_2m) \text{ for all } n_1, n_2\in N, \ m\in M.$$

If there is no danger of confusion, we will speak of the module M instead of the module ${}_NM$ and if we wish to emphasize the near-ring N, we will speak of N-module M.

Let M be a N-module. Then a subgroup A of M is called a *N-subgroup of M* if

$$NA=\{na\,|\,n\in N,\ a\in A\}\subseteq A.$$

Let A be a subset of a N-module M. A is called a *N-submodule of*

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M if it is a normal subgroup of M and if for all $n\in N$, $m\in M$, $a\in A$, $n(m+a)-nm\in A$. Clearly if A is a N-subgroup (left ideal) of a near-ring N, then A may be regarded as N-subgroup (N-submodule) of ${}_NN$.

Let M be a near-ring module and let H be a submodule of M and L a N-subgroup of M. Then $H+L=\{x+y\mid x\in H, y\in L\}$ is a N-subgroup of M and $H+L=L+H$. Moreover if L is a submodule, then $H+L$ is a submodule of M.

The left ideal generated by $A\subseteq N$ is denoted by $(A|$, and the submodule generated by $B\subseteq_N M$ is denoted by $(B|$. Let M be a N-module. Then a subset T of M is called a *full generator* (*fg*) *for* M if every submodule of M is generated by a subset of T.

Notation and terminology not defined here follow Pilz [6].

3. Characterizations of prime near-rings

DEFINITION 1. An ideal P of a near-ring N is called *prime* if $AB\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$, for any ideals A and B of N.

N is called a *prime near-ring* if (0) is a prime ideal.

Every integral near-ring is a prime near-ring and each constant near-ring is a prime near-ring. Of course N is a prime ideal of N, so (0) is a prime near-ring.

DEFINITION 2. Let M be a N-module. Then a submodule B of M is called *prime* (*weakly prime*) if $LB'\subseteq B$ implies $B'\subseteq B$, for any nonzero left ideal L of N and N-subgroup (submodule) B' of M.

M is called a *prime near-ring module* if (0) is a prime submodule of M.

DEFINITION 3. A left ideal L of a near-ring N is called *prime* if L is a prime submodule of ${}_NN$.

N is called a *strictly prime near-ring* if ${}_NN$ is prime.

REMARK 1. Let A be a nonzero ideal of a near-ring N. Then A can not be contained in a prime left ideal $I\neq N$. For $AN\subseteq A\subseteq I$ implies that $N\subseteq I$. Hence the concept of a prime left ideal has no content in a commutative near-ring.

REMARK 2. Even a noncommutative near-ring N must be a prime near-ring if it is to have any prime left ideal other than N. For let $AB=(0)$, A and B nonzero ideals of N, then $B\subseteq I$, for any prime left ideal I of N. Hence $I=N$, by our previous remark 1.

REMARK 3. Let M be a prime near-ring module. Then both (0) and M are prime submodules of M.

Every constant near-ring is a prime near-ring but not strictly prime near-ring. If a near-ring N with identity has no nonzero proper left ideal of N, then a submodule of a unitary N-module M is prime.

If a near-ring N can be regarded as a near-ring module. Then many of the result that follow on prime submodules of ${}_NN$ give theorems on the prime left ideals of N.

THEOREM 4 [6, 2.95]. *Let N be a zero symmetric near-ring with identity. Then if ${}_NM$ is a prime near-ring module, then N is a prime near-ring (but not conversely). In particular, if N is a strictly prime near-ring, then N is a prime (but not conversely even if N is finite).*

THEOREM 5. *Let N be a zero symmetric near-ring, M_1 be a N-subgroup of a N-module M such that M_1 is a fg for M, and let Q be a submodule of M. If $M_1 \cap Q$ is a weakly prime submodule in M_1, then Q is a weakly prime submodule in M.*

Proof. Suppose $P=M_1\cap Q$ is weakly prime in M_1. Let L be any nonzero left ideal of N and A a submodule of M such that $LA\subseteq Q$. Then $L(A\cap M_1) \subseteq LA\cap M_1\subseteq Q\cap M_1=P$. Since $P=M_1\cap Q$ is a weakly prime submodule in M_1, $A\cap M_1\subseteq P$. Now M_1 is a fg for M, $A=(A\cap M_1|\subseteq(M_1\cap Q|=Q$. Thus Q is a weakly prime submodule in M.

THEOREM 6. *Let $\{A_\alpha | \alpha\in I\}$ be a family of prime submodules of M. Then $\bigcap_{\alpha\in I} A_\alpha$ is again prime. Thus every submodule A of M has a unique prime cover $P(A)$ consisting of the intersection of all prime submodules containing A.*

Proof. Let $LA\subseteq\bigcap_{\alpha\in I} A_\alpha$, A a N-subgroup of M and L a nonzero left ideal of N. Then $LA\subseteq A_\alpha$ for all $\alpha\in I$. Since A_α is a prime submodule of M, $A\subseteq A_\alpha$ for all $\alpha\in I$. Hence $A\subseteq\bigcap_{\alpha\in I} A_\alpha$.

LEMMA 7 [6, Theorem 2.20]. *Let M be a N-module and $L(M)$ be the set of all submodules of M. Then $L(M)$ is a complete (modular) lattice.*

DEFINITION 8. A *closure operation* on a complete lattice $(S, \leq)$ is understood a mapping $a \to a^c$ of S into S such that $a \leq a^c$, $(a^c)^c \leq a^c$, $a \leq b \Rightarrow a^c \leq b^c$.

An element a of S is called *closed* if $a^c = a$.

THEOREM 9. *Let $L(M)$ be the complete lattice of all submodules of M. Then the mapping $A \to P(A)$ is a closure operation on $L(M)$.*

Proof. Let $A \in L(M)$, then there exists a unique prime cover $P(A)$ of A. Hence $A \subseteq P(A)$ and $P(A) = P(P(A)))$. If $A \subseteq B$, then $P(A) \subseteq P(B)$. Thus $A \to P(A)$ is a closure operation on $L(M)$.

If A and B are submodules of the prime near-ring module M such that $A \cap B = (0)$. Let

$S = \{B' \mid B'$ is a submodule of M such that $A \cap B' = (0)$ and $B' \supseteq B\}$. Then $S \neq \phi$, for $B \in S$, and $(S, \subseteq)$ is an ordered set. Hence Zorn's lemma allows us to find a maximal element B'' in S. Such a submodule as B'' is called a *complement of A* (*over B*). Any left ideal L of N for which $L \cap L_1 = (0)$ has a complement $L'(\supseteq L_1)$ as a consequence.

THEOREM 10. *Let A be a submodule of the prime distributive near-ring module M and A' be a complement of A. Then A' is a prime submodule of M.*

Proof. Suppose that $LA_1 \subseteq A'$, A_1 a N-subgroup of M and L a nonzero left ideal of N. Then $L[(A_1 + A' \mid \cap A] \subseteq A' \cap A = (0)$, and since M is prime, $(A_1 + A' \mid \cap A = (0)$. Now A' is a complement of A, and therefore $(A_1 + A' \mid \subseteq A'$ Thus $A_1 \subseteq A'$ and A' is a prime submodule of M.

DEFINITION 11. Let M be a N-module and let M_1, M_2 be subsets of M. Then $(M_1 : M_2) = \{n \in N \mid nM_2 \subseteq M_1\}$, we denote $(\{m\} : M_2) = (m : M_2)$ for $m \in M$, similarly for $(M_1 : \{m\}) = (M_1 : m)$. Moreover $(0 : M_2)$ is called the *left annihilator* of M_2.

LEMMA 12. *Let A be a submodule of ${}_NM$ and A_1 be a subset of M. Then $(A : A_1)$ is a left ideal of N and $(A : A_1) = \bigcap_{x \in A_1} (A : x)$. In particular, $(0 : m)$ is a left ideal of N.*

Proof. See [6, p. 21].

THEOREM 13. *Let A be a prime submodule of the prime near-ring module ${}_NM$ and let A_1 be a subset of M. Then $(A:A_1)$ is a prime left ideal of N.*

Proof. Let $x \in A_1$ and let $L_1L_2 \subseteq (A:x)$ for L_1 a nonzero left ideal of N, L_2 a N-subgroup of N. Then $(L_1L_2)x = L_1(L_2x) \subseteq A$. Since A is a prime submodule of M, $L_2x \subseteq A$. Hence $L_2 \subseteq (A:x)$. Therefore $(A:x)$ is a prime left ideal of N. Since $(A:A_1) = \bigcap_{x \in A_1}(A:x)$, $(A:A_1)$ is a prime left ideal of N.

COROLLARY 14. *Let ${}_NM$ be a prime near-ring module and $x \in M$. Then the left annihilator $(O:x)$ is a prime left ideal of N.*

THEOREM 15. *For any x in the prime near-ring module ${}_NM$ and left ideal L of N, $P(L)x \subseteq P((Lx|)$.*

Proof. Since $Lx \subseteq (Lx| \subseteq P(Lx|)$, $L \subseteq (P((Lx|):x)$. Since $(P((Lx|):x)$ is prime left ideal of N, $P(L) \subseteq P((Lx|):x)$. Hence $P(L)x \subseteq P((Lx|)$.

Hereafter N is a distributive near-ring. Let L be a left ideal of a distributive near-ring N and let $L^r = \{n \in N \mid Ln = 0\}$ be the right annihilator of L. Then L^r is a right ideal of N. For if $n \in L^r$, $x \in N$, $l \in L$, then $l(x+n-x) = lx + ln - lx = 0$. So $L(x+n-x) = 0$. Thus L^r is normal and $(Ln)N = L(nN) = 0$. Hence L^r is a right ideal of N.

Let I be a right ideal of N and let $I^l = \{n \in N \mid nI = 0\}$ be the left annihilator of I. Then I^l is a left ideal of N. In particular, if N is strictly prime, then I^l is a prime left ideal of N.

LEMMA 16. *Let I and I' be right ideals of N and let L and L' be left ideals of N. Then*

(1) *$I \subseteq I^{lr}$ and $L \subseteq L^{rl}$*

(2) *If $I' \subseteq I$, then $I^l \subseteq I'^l$*

(2′) *If $L' \subseteq L$, then $L^r \subseteq L'^r$*

(3) *$I^{lrl} = I^l$ and $L^{rlr} = L^r$*

Proof. (1) Since $I^lI = 0$, $I \subseteq I^{lr}$.
Similarly for $L \subseteq L^{rl}$.

(2) It is obvious.

(3) By (1), $I^l \subseteq I^{lrl}$ and, by (2), $I^{lrl} \subseteq I^l$. Thus $I^l = I^{lrl}$.

For any left ideal L and right ideal I of N, let us define $l(L) = L^{rl}$ and $r(I) = I^{lr}$. Denote by $\mathscr{U}_l$ the set of all left ideals L such that $L = l(L)$, and $\mathscr{U}_r$ the set of all right ideals I such that $I = r(I)$.

THEOREM 17. *The mapping $L \mapsto l(L)$ is a closure operation on the complete lattice of all left ideals of N.*

Proof. It is immediate from Lemma 16.

THEOREM 18. *The lattices $\mathscr{U}_r$ and $\mathscr{U}_l$ are anti-isomorphic under the correspondence $I \mapsto I^l$, $I \in \mathscr{U}_r$.*

Proof. Let I, $I' \in \mathscr{U}_r$, then $I \subseteq I + I'$ and $I' \subseteq I + I'$. Thus $(I+I')^l \subseteq I^l \cap I'^l$. Now let $x \in I^l \cap I'^l$, then $xI = 0$ and $xI' = 0$. Hence $x(I+I') = xI + xI' = 0$ and $x \in (I+I')^l$. Therefore $(I+I')^l = I^l \cap I'^l$. Thus $I \cap I' = I^{lr} \cap I'^{lr} = (I^l + I'^l)^r \in \mathscr{U}_r$. Define $I \cup I' = r(I+I')$ in $\mathscr{U}_r$. It is clear that $\mathscr{U}_r$ and $\mathscr{U}_l$ are lattices.

Define $f : \mathscr{U}_r \to \mathscr{U}_l$ by $I \mapsto I^l$.

It is clear that f is bijective, and that $f(I' \cup I) = (I' + I)^l = I'^l \cap I^l = f(I') \cap f(I)$, $f(I' \cap I) = f(I'^{lr} \cap I^{lr}) = f((I'^l + I^l)^r) = (I'^l + I^l)^{rl} = I'^l \cup I^l = f(I') \cup f(I)$. Thus f is a lattice anti-isomorphism.

THEOREM 19. *Let N be a near-ring and let L be a nonzero left ideal of N such that left $ann(L) = \{n \in N \mid nL = 0\} = (0)$. Then N is commutative if L is.*

Proof. Assume that L is commutative. Let $n \in N$, $x \in L$ be given. Then for any $y \in L$,

$$\begin{aligned}(nx - xn)y &= (nx + (-(xn)))y = (nx)y + ((-x)n)y \\ &= n(xy) + (-x)(ny) = (ny)x + (-x)(ny) \\ &= x(ny) + (-x)(ny) = (x + (-x))ny = 0.\end{aligned}$$

Since $y \in L$ is arbitrary, $nx - xn \in$ left $\mathrm{ann}(L) = (0)$. Hence $nx = xn$. Thus L is contained in the center of N. Next suppose that n, $s \in N$ and let $x \in L$. Then $(ns - sn)x = n(sx) - s(nx) = (sx)n - s(xn) = 0$, for $sx \in L$. Hence $ns - sn \in$ left ann $(L) = (0)$, n and s being arbitrary. So the proof is complete.

LEMMA 20. *Let N be a distributive near-ring. Then N is either each* *element is a zero-divisor or N is a ring.*

Proof. It is a result of Taussky. See [6]. p. 332.

THEOREM 21. *Let N be a near-ring without zero-divisor. Suppose N has a nonzero commutative left ideal L. Then N is a commutative ring.*

Proof. It follows from Theorem 19 and Lemma 20.

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