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REMARKS ON BCI-ALGEBRAS

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In 1966, K. Iseki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In this paper we prove that $HOM_*(X, Y)$ is an associative BCI-algebra in which every element is of order two. Also we introduce the notion of an exact sequence of homomorphisms of BCI-algebras, and give some related results.

Definition 1. [2] A *BCI-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ with the following conditions:

(I) $(x*y)*(x*z)\leq z*y$,
(II) $x*(x*y)\leq y$,
(III) $x\leq x$,
(IV) $x\leq y$ and $y\leq x$ imply $x=y$,
(V) $x\leq 0$ implies $x=0$,

where $x\leq y$ if and only if $x*y=0$.

It is proved by Iseki [4] that in any BCI-algebra X we have

$$(x*y)*z=(x*z)*y \text{ and } x*0=x$$

for all x, y and z in X.

Definition 2. [5] A mapping $f: X\to Y$ between BCI-algebras X and Y is called a *homomorphism* if

$$f(x*y)=f(x)*f(y)$$

for all $x, y\in X$.

LEMMA 3. *Let X, Y and Z be BCI-algebras, and let $h: X\to Y$ be an epimorphism, and let $g: X\to Z$ be a homomorphism. If $Ker(h)\subset Ker(g)$,*

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then there is a unique homomorphism $f: Y \to Z$ *satisfying* $fh = g$.

Proof. The proof is quite similar to the case of a BCK-algebra (see [3], p. 22).

LEMMA 4. *Let X, Y and Z be BCI-algebras, and let* $g: X \to Z$ *be a homomorphism, and let* $h: Y \to Z$ *be a monomorphism with* $Im(g) \subset Im(h)$. *Then there is a unique homomorphism* $f: X \to Y$ *satisfying* $g = hf$.

Proof. For each $x \in X$, $g(x) \in \mathrm{Im}(g) \subset \mathrm{Im}(h)$. Since h is a monomorphism, there exists a unique $y \in Y$ such that $h(y) = g(x)$. Therefore there is a function $f: X \to Y$, $x \mapsto y$, such that $hf = g$. To show that f is a homomorphism, let $x_1, x_2 \in X$, then

$$g(x_1 * x_2) = h(f(x_1 * x_2)).$$

On the other hand, since g is a homomorphism,

$$\begin{aligned} g(x_1 * x_2) &= g(x_1) * g(x_2) \\ &= h(f(x_1)) * h(f(x_2)) \\ &= h(f(x_1) * f(x_2)). \end{aligned}$$

Hence $h(f(x_1 * x_2)) = h(f(x_1) * f(x_2))$. Since h is a monomorphism, $f(x_1 * x_2) = f(x_1) * f(x_2)$. The uniqueness of f is trivial since h is a monomorphism.

Definition 5. A sequence of homomorphisms of BCI-algebras

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called a *zero*(resp. an *exact*) *sequence* if $gf = 0$ (resp. $\mathrm{Im}(f) = \mathrm{Ker}(g)$).

THEOREM 6. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence of homomorphisms of BCI-algebras, and let* $h: Y \to A$ *be a homomorphism of BCI-algebras such that* $hf = 0$. *Then there is a unique homomorphism* $\phi: Z \to A$ *satisfying* $\phi g = h$.

Proof. Since $hf = 0$, we have $\mathrm{Ker}(g) = \mathrm{Im}(f) \subset \mathrm{Ker}(h)$. It follows from the Lemma 3 that there is a unique homomorphism $\phi: Z \to A$ satisfying $\phi g = h$.

THEOREM 7. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ *be an exact sequence of homomorphisms* *of BCI-algebras, and let* $h: A \to Y$ *be a homomorphism of BCI-algebras such that* $gh=0$. *Then there is a unique homomorphism* $\psi: A \to X$ *satisfying* $f\psi=h$.

Proof. Since $gh=0$, we have $\mathrm{Im}(h) \subset \mathrm{Ker}(g) = \mathrm{Im}(f)$. It follows from the Lemma 4 that there is a unique homomorphism $\psi: A \to X$ satisfying $f\psi=h$.

Definition 8. [5] A BCI-algebra X is said to be *associative* if $(x*y)*z = x*(y*z)$ for all x, y and z in X.

Example. Let X be a trivial BCK-algebra, and let a be an ideal element. We define

$$0*a=a*0=a,$$
$$a*a=0.$$

Then $X \cup \{a\}$ is a BCI-algebra, but not a BCK-algebra. Moreover, $X \cup \{a\}$ is an associative BCI-algebra.

Let X and Y be BCI-algebras and let $\mathrm{Hom}_*(X, Y)$ denote the set of all homomorphisms with an associative BCI-algebra as codomain. Let $f, g \in \mathrm{Hom}_*(X, Y)$. Define a mapping $f*g: X \to Y$ by

$$(f*g)(x) = f(x)*g(x)$$

for all $x \in X$. Then we have the following:

THEOREM 9. *Let* X *and* Y *be BCI-algebras and* $f, g \in Hom_*(X, Y)$. *Then* $f*g \in Hom_*(X, Y)$.

Proof. For any $x, y \in X$,

$$\begin{aligned}(f*g)(x*y) &= f(x*y)*g(x*y) \\ &= (f(x)*f(y))*(g(x)*g(y)) \\ &= (f(x)*(g(x)*g(y)))*f(y) \\ &= ((f(x)*g(x))*g(y))*f(y) \\ &= ((f(x)*g(x))*f(y))*g(y) \\ &= (f(x)*g(x))*(f(y)*g(y)) \\ &= (f*g)(x)*(f*g)(y).\end{aligned}$$

This implies $f*g \in \mathrm{Hom}_*(X, Y)$.

For any $f, g, h \in \mathrm{Hom}_*(X, Y)$ and $x \in X$, we have

$$\begin{aligned}((f*g)*h)(x) &= (f*g)(x)*h(x) \\ &= (f(x)*g(x))*h(x) \\ &= f(x)*(g(x)*h(x)) \\ &= f(x)*(g*h)(x) \\ &= (f*(g*h))(x),\end{aligned}$$

and

$$(f*f)(x) = f(x)*f(x) = 0.$$

It follows that $(f*g)*h = f*(g*h)$ and $f*f = 0$, the zero homomorphism. Hence we have the following:

THEOREM 10. *Let X, Y be BCI-algebras. Then $Hom_*(X, Y)$ with the composition $*$ and the constant 0 given by $0 : x \mapsto 0$ is an associative BCI-algebra in which every element is of order two.*

The following is easily verified:

THEOREM 11. *For $f, f' \in Hom_*(X, Y)$ and $g, g' \in Hom_*(Y, Z)$, the following distributive laws hold:*

(1) $g(f*f') = (gf)*(gf')$.
(2) $(g*g')f = (gf)*(g'f)$.

Let X, Y and Z be BCI-algebras. For a fixed element f in $\mathrm{Hom}_*(X, Y)$, we define maps

$$f_* : \mathrm{Hom}_*(Z, X) \to \mathrm{Hom}_*(Z, Y)$$

and

$$f^* : \mathrm{Hom}_*(Y, Z) \to \mathrm{Hom}_*(X, Z)$$

by

$$f_*(\psi) = f\psi \text{ and } f^*(\phi) = \phi f$$

for all $\psi \in \mathrm{Hom}_*(Z, X)$ and all $\phi \in \mathrm{Hom}_*(Y, Z)$.

Clearly f_* and f^* are homomorphisms of BCI-algebras.

LEMMA 12. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a zero sequence of homomorphisms of BCI-algebras.*

(1) *Assume that for every homomorphism* $u: M \to Y$ *of BCI-algebras with* $gu=0$ *there is exactly one homomorphism* $v: M \to X$ *of BCI-algebras with* $u=fv$. *Then the sequence*

$$Hom_*(M, X) \xrightarrow{f_*} Hom_*(M, Y) \xrightarrow{g_*} Hom_*(M, Z)$$

is exact.

(2) *Assume that for every homomorphism* $u: Y \to N$ *of BCI-algebras with* $uf=0$ *there is exactly one homomorphism* $v: Z \to N$ *of BCI-algebras with* $u=vg$. *Then the sequence*

$$Hom_*(Z, N) \xrightarrow{g^*} Hom_*(Y, N) \xrightarrow{f^*} Hom_*(X, N)$$

is exact.

Proof. (1) Since $gf=0$, we have $g_*f_*(v)=gfv=0$. It follows that $\mathrm{Im}(f_*) \subset \mathrm{Ker}(g_*)$. Let $u \in \mathrm{Ker}(g_*)$. Then $g_*(u)=gu=0$, and hence by the assumption there exists a homomorphism $v: M \to X$ such that $u=fv$. This means that $f_*(v)=u$. Hence $u \in \mathrm{Im}(f_*)$. Thus we have $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$.

(2) Since $gf=0$, we have $f^*g^*(v)=f^*(vg)=v(gf)=0$. This shows $\mathrm{Im}(g^*) \subset \mathrm{Ker}(f^*)$. Let $u \in \mathrm{Ker}(f^*)$. Then $f^*(u)=uf=0$. By the assumption, there exists a homomorphism $v: Z \to N$ satisfying $u=vg$. It follows that $u=g^*(v)$. Hence $u \in \mathrm{Im}(g^*)$, which completes the proof.

THEOREM 13.

(1) *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ *be an exact sequence of homomorphisms of BCI-algebras. For every BCI-algebra M, the sequence*

$$0 \to Hom_*(M, X) \xrightarrow{f_*} Hom_*(M, Y) \xrightarrow{g_*} Hom_*(M, Z)$$

is exact.

(2) *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence of homomorphisms of BCI-algebras. For every BCI-algebra N the sequence*

$$0 \to Hom_*(Z, N) \xrightarrow{g^*} Hom_*(Y, N) \xrightarrow{f^*} Hom_*(X, N)$$

is exact.

Proof. (1) By Theorem 7 and Lemma 12, the sequence

$$Hom_*(M, X) \xrightarrow{f_*} Hom_*(M, Y) \xrightarrow{g_*} Hom_*(M, Z)$$

is exact. Next we show that the sequence

$$0 \rightarrow Hom_*(M, X) \xrightarrow{f_*} Hom_*(M, Y)$$

is exact. Let $v \in \mathrm{Ker}(f_*)$. Then $fv=0$. Since f is injective, we have $v=0$. It follows that the sequence is exact.

(2) By Theorem 6 and Lemma 12, the sequence

$$\mathrm{Hom}_*(Z, N) \xrightarrow{g^*} Hom_*(Y, N) \xrightarrow{f^*} Hom_*(X, N)$$

is exact. Next we show that the sequence

$$0 \rightarrow \mathrm{Hom}_*(Z, N) \xrightarrow{g^*} \mathrm{Hom}_*(Y, N)$$

is exact. Let $w \in \mathrm{Ker}(g^*)$. Then $wg=0$, and hence $w=0$ by the surjectivity of g. It follows that the sequence is exact.

References

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