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NONLINEAR HARMONIC ANALYSIS

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Classical harmonic analysis concerns the study of harmonic functions on a domain. Especially, the boundary behavior of harmonic functions is one of the central issues of harmonic analysis because of its direct relationship with Fourier series. Harmonic analysis has a long and rich history, and it by now possesses a well defined body of knowledge. (See, for example, [SW].) However, the whole theory is distinctively linear, and it was not clear how to extend this linear theory to nonlinear situation. Recently, however, P. Aviles, M. Micallef, and the author have succeeded in extending the harmonic analysis results to harmonic maps, which is a natural nonlinear generalization of harmonic function [ACM]. Harmonic map arises naturally from differential geometry and physics (σ-model, liquid crystal, etc.), and has been the object of much interest from differential geometry, nonlinear elliptic partial differential equation, and physics. In this paper, we will first describe the recent development from the perspective of harmonic analysis, and then show how our results extend naturally to most general situation. The author would like to thank professor Dong Pyo Chi for his interest in our work.

§1. Harmonic Map

Let M and N be Riemannian manifolds of dimension m and n respectively. Let $x^1, x^2, \cdots, x^m$ be local co-ordinates of M, and $y^1, y^2, \cdots, y^n$ those of N. Let $g=g_{\alpha\beta}dx^\alpha dx^\beta$ and $h=h_{ij}dy^i dy^j$ be the metrics of M and N, respectively. The repeated indices are summed; Greek indices range from 1 to m, and Latin 1 to n. Let $u: M \rightarrow n$ be a map. In terms of local co-ordinates, $y^1, y^2, \cdots, y^n$, u is represented by an n-tuple of functions $(u^1, \cdots, u^n)$. We define the energy density $e(u)=tr_g(u^*h)=g^{\alpha\beta}h_{ij}\frac{\partial u^i}{\partial x^\alpha}$

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$\frac{\partial u^j}{\partial x^\beta}$. Harmonic map is a critical point of the functional $\int_M e(u)dv$. In terms of local co-ordinates, harmonic map satisfies the following nonlinear elliptic system of PDE.

$$\Delta u^i + \Gamma_{jk}{}^i(u) g^{\alpha\beta} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0, \quad \text{for } i=1, 2, \cdots, n,$$

where $\Gamma_{jk}{}^i$ is the Christoffel symbol of N.

If N is flat, all Christoffel symbols vanish, and the harmonic map equation becomes the Laplace-Beltrami equation. In other words, harmonic map is a nonlinear generalization of harmonic function, and the nonlinearity is due to the presence of curvature in the target manifold N.

It is well known from the linear theroy that harmonic function is smooth in the interior of the domain of definition. However, harmonic map, in general, has singularity, (See [SU] for singularity estimates for energy minimizing harmonic maps.) It is therefore clear that the existence of singularity must be due to *nonlinearity*. In fact, there has been much activity to try to understand the size and structure of singularity. The nonlinearity of harmonic map is at the lower order term, and the leading second order part is linear, which makes the nonlinearity rather mild, and in fact it was suspected that the nonlinear terms should play relatively minor role, if the harmonic map is smooth. We can therefore ask the following general question: How close is harmonic map to harmonic function when it is smooth? We found the answer in the affirmative in the sense given below. For the clearity of presentation, we assume, in §2 and §3, that M is a bounded domain in $R^m (m \geq 3)$ with flat metric. However, it is easy to see that the results hold for general Riemannian manifold with minor modification, and for $m=2$.

§2. Wiener Criterion and Dirichlet Problem on Nonsmooth Domain

The solution of Dirichlet problem for harmonic function is one of the fundamental results in classical harmonic analysis.

A very flexible method was developed by Perron, and N. Wiener obtained the necessary and sufficient condition for the solution to take on the given boundary value [W].

DEFINITION. Let Ω be a bounded domain in $\boldsymbol{R}^m$. A point $p \in \partial\Omega$ is called a regular point (for Dirichlet problem), if, for any continuous boundary data $\phi : \partial\Omega \to \mathbf{R}$, there exists $h : \Omega \to \mathbf{R}$ such that $\Delta h = 0$ on Ω and $h(x) \to \phi(p)$ as $x \to p$.

DEFINITION. Let K be a compact subset in $\mathbf{R}^m$, the capacity cap(K) of K is defined by

$$\mathrm{cap}(K) = \inf\left\{\int_{\mathbf{R}^m \cap K} |\nabla f|^2 dv \,|\, f \in C_0^1(\mathbf{R}^m) \text{ and } f \equiv 1 \text{ on } K\right\}$$

WIENER CONDITION. Let Ω be a bounded domain in $\mathbf{R}^m$. Let $p \in \partial\Omega$. Let σ be a real number such that $0 < \sigma < 1$, and j a positive integer. Define $x(\sigma^j) = \sigma^{(2-n)j} \mathrm{cap}\{B_{\sigma^j}(p) \backslash B_{\sigma^{j+1}}(p) \backslash \Omega\}$. p is called a Wiener point, if $\sum_{j=1}^{\infty} \chi(\sigma^j)$ diverges for some $\sigma \in (0, 1)$.

THEOREM (Wiener Criterion). *$p \in \partial\Omega$ is regular if and only if p is a Wiener point.*

This theorem reduces the test for regularity to a computable (at least in principle) formula. The Dirichlet problem for harmonic map was first solved by R. Hamilton [H] when the target manifold has nonpositive curvature, and by S. Hildebrandt, H. Kaul, and K.-O. Widman [HKW] when the target manifold has positive curvature. In either case the solution is obtained for the harmonic map lying in the geodesic ball $B_R(q)$ such that R is bounded if N is a simply connected complete manifold of nonpositive curvature, and if N has positive curvature, $R < \min\left\{\frac{\pi}{2\sqrt{k}}, i(p)\right\}$ where $i(p)$ is the injectivity radius at p, and k the upper bound of sectional curvature of N. For convenience, let us call the above condition HKW condition. It is perhaps relevant to make note that when N has positive curvature, the above assumption is optimal in the sense that if $R \geq \min\left\{\frac{\pi}{2\sqrt{k}}, i(p)\right\}$, the solution may have singularity. As we mentioned in §1, we want to restrict ourselves to the situation when the solution is guaranteed to be smooth. The results of Hamilton and Hildebrandt-Kaul-Widman are PDE results, and because the proof involves boundary estimates, boundary must be assumed to have certain regularity, say $C^{1,\alpha}$, and the boundary data must be C^{α}. In [ACM], a method to avoid the boundary estimates altogether was developed, and as a consequence, the following result, which is as sharp as the harmonic analysis result of Wiener, is obtained.

THEOREM. *Let $\Omega\subset\mathbf{R}^m$ be a nonsmooth bounded domain. Suppose $\phi:\partial\Omega\to B_R(q)$ be a continuous map. Assume $B_R(q)$ satisfies the HKW condition. Then there exists a harmonic map $u:\Omega\to B_R(q)$ such that $u(x)\to\phi(p)$ as $x\to p$ for every Wiener point $p\in\partial\Omega$.*

Sketch of Proof. $\phi=(\phi^1,\phi^2,\cdots,\phi^n)$ is an n-tuple of functions. Let $h=(h^1,h^2,\cdots,h^n)$ be the harmonic extension of ϕ such that h^j is the solution of the Dirichlet problem for the boundary data ϕ^j. Let $\{\Omega_i\}$ be a family of smooth subdomains of Ω such that $\bigcup_i\Omega_i=\Omega$ and $\Omega_i\subset\subset\Omega$. Since Ω_i is smooth, one can obtain the harmonic map $u_i:\Omega_i\to B_R(q)$ such that $u_i=h$ on $\partial\Omega_i$. We need to estimate the distance between u_i and h in the C^0 sense. Let $\rho(y,z)$ be the geodesic distance in N between two points y and z of N. Define

$$A(y,z)=\begin{cases}\frac{1}{k}(1-\cos(\sqrt{k}\,\rho(y,z))) & \text{if } k>0,\\ \frac{1}{2}\rho^2(y,z) & \text{if } k=0.\end{cases}$$

We define a function $\psi(x)$ on Ω by

$$\psi(x)=\frac{A(u_i(x),\ h(x))}{\cos(\sqrt{k}\,\rho(u_i(x),q))}.$$

A lengthy computation shows that $\Delta\psi\geq-C|\nabla h|^2=-\frac{C}{2}|\Delta h|^2$, where C is a constant depending only on the geometry of Ω and $B_R(q)$, and $|\nabla h|^2=\sum_{i=1}^{n}|\nabla h^i|^2$, and $|h|^2=\sum_{i=1}^{n}|h^i|^2$. Let v_i be a harmonic function on Ω_i such that $v_i=\psi+\frac{C}{2}|h|^2$ on $\partial\Omega_i$. By maximum principle, we have $\psi+\frac{C}{2}|h|^2\leq v_i$. Thus $\psi\leq v_i-\frac{C}{2}2|h|^2$. Since $\psi=0$ on $\partial\Omega_i$, $v_i-\frac{C}{2}|h|^2=0$ on $\partial\Omega_i$. Therefore we have the following C^0 estimate

$$\rho^2(u_i(x),\ h(x))\leq C_1(v_i(x)-\frac{C}{2}|h(x)|^2),\ \text{for all } x\in\Omega_i,$$

where C_1 is a universal constant.

By the gradient estimate of the author [C1], the sequence $\{u_i\}$ has a subsequence converging to a harmonic map u, and $\{v_i\}$ also has a subsequence converging to a harmonic function v. Therefore we have

$$\rho^2(u(x),\ h(x)) \leq C_1((v(x) - \frac{C}{2}|h(x)|^2) \text{ for all } x \in \Omega.$$

Since v and $\frac{C}{2}|h|^2$ have the same boundary value at every Wiener point, u must have the same boundary value as h has at p, which is $\phi(p)$.

REMARK. One can modify the above statement and proof to allow ϕ to be measurable, which is of interest in itself.

§3. Fatou's Theorem

Another interesting result from harmonic analysis is Fatou's theorem which states as follows: Let D be the open unit disk in C, and let h be a positive harmonic function on D. Then $\lim_{r \to 1} h(re^{i\theta}) = \phi(\theta)$ exists for almost all θ. One can actually allow the limit to be *nontangential limit*. The significance of Fatou's theorem is the following: By the Poisson formula,

$$h(x) = \int_{S^1} K(x, \theta) d\mu(\theta),$$

where $d\mu$ is a measure representing the "boundary value" of h. With respect to Lebesgue measure $d\theta$ of the boundary, $d\mu$ can be written as $d\mu = d\mu_{a.c.} + d\mu_{sing}$, where $d\mu_{a.c.}$ is the absolutely continuous part of $d\mu$ w.r.t. $d\theta$, and $d\mu_{sing}$ the singular part. Let $\frac{d\mu_{a.c.}}{d\theta}$ be the Radon-Nikodym derivative of $d\mu_{a.c.}$ with respect to $d\theta$, Then $\phi = \frac{d\mu_{a.c.}}{d\theta}$. In other words, the nontangential limit recovers the absolutely continuous part of $d\mu$. If, in addition, h is bounded, then $d\mu_{sing} = 0$. Therefore the nontangential limit completely recovers the *bounded* harmonic function h, and in terms of the Poisson formula

$$h(x) = \int_{S^1} K(x, \theta) \phi(\theta) d\theta.$$

The situation for solutions of nonlinear elliptic equation is quite different, and in general Fatou-type results do not hold. However, our C^0 estimate is a rather powerful tool to control the boundary behavior, and in fact we obtain the following Fatou-type theorem.

THEOREM. *Let $\Omega \subset \boldsymbol{R}^m$ be a bounded Lipschitz domain. Suppose $u : \Omega \to B_R(q)$ is a harmonic map defined in the interior of Ω, and $B_R(q)$ is assumed to satisfy HKW condition. Then the nontangential limit exists for almost all $Q \in \partial\Omega$.*

Sketch of Proof. Let $\{\Omega_i\}$ be subdomains as in the proof of Theorem in §2. Let $h_i = (h_i^1, h_i^2, \cdots, h_i^n)$ be an n-tuple of harmonic functions such that $h_i = u$ on $\partial\Omega_i$. Apply the argument for C^0-estimate to obtain $\rho^2(u, h_i) \leq C_1\left(v_i - \frac{C}{2}|h_i|^2\right)$. By the standard gradient estimate, $\{h_i\}$ has a convergent subsequence. Therefore, we have $\rho^2(u, h) \leq C_1\left(v - \frac{C}{2}|h|^2\right)$. A subtle argument then shows that $v - \frac{C}{2}|h|^2$ achieves zero boundary value almost everywhere on $\partial\Omega$. One can then apply usual Fatou's theorem for harmonic functions, which says that h has nontangential limit ϕ almost everywhere on $\partial\Omega$. Combining these two statements, the proof is complete.

REMARK (Recovering u from ϕ). One can in fact recover u from ϕ. Let ϕ be the nontangential limit obtained by the above theorem. Let $h = (h^1, h^2, \cdots, h^n)$ be the n-tuple of harmonic functions with "boundary value" ϕ. Let $\{\Omega_i\}$ be the subdomains as before, and let $u_i : \Omega_i \to B_R(q)$ be a harmonic map such that $u_i = h$ on $\partial\Omega_i$. Then by the argument similar to the one given above, $\{u_i\}$ has a subsequence convergent to a harmonic map with "boundary value" ϕ. By the maximum principle, this new harmonic map must be the original u.

§4. Nonlinear Harmonic Analysis on Complete Manifold

Let M be a complete, simply connected Riemannian manifold with sectional curvature K_M satisfying $-b^2 \leq K_M \leq -a^2 < 0$. Two unit speed geodesics γ_1 and γ_2 are called asymptotic, if $\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) \leq C$ for some constant C. The set of equivalence classes of asymptotic geodesics is called the sphere at infinity, and is denote by $S(\infty)$. Intuitively,

$S(\infty)$ is analogous to the extrinsic boundary $\{x \mid |x|=1\}$ when M is the unit ball $\{x \mid |x|<1\}$ with the Poincaré metric. We can therefore pose the Dirichlet problem for harmonic function on M with $S(\infty)$ as the boundary. The author first studied this problem and obtained a sufficient condition for the solvability of Dirichlet problem in terms of certain convexity condition at infinity [C2]. This condition was later verified by M. Anderson under the current curvature assumption [A]. Later, M. Anderson and R. Schoen produced many interesting harmonic analysis results on M. Among other things, they identified the Martin boundary, and wrote down the Poisson formula, and proved the Fatou's theorem.

In this context, our previous results from §2 and §3 are still valid, although some nontrivial work is needed.

THEOREM. *Let $\phi : S(\infty) \to B_R(q)$ be a continuous map. Then there exists a harmonic map $u : M \to B_R(q)$ such that $u|_{S(\infty)} = \phi$.*

THEOREM. *Let $u : M \to B_R(q)$ be any harmonic map. Then the nontangential limit exists for almost all $Q \in S(\infty)$.*

Furthermore, if ϕ is C^α, one can obtain much detailed information on u.

THEOREM. *Let $\phi : S(\infty) \to B_R(q)$ be a C^α map. Then there exists a harmonic map $u : M \to B_R(q)$ such that $\rho(u(x), \phi(x)) \le Ce^{-\delta r(x)}$, and $e(u)(x) \le Ce^{-\delta r(x)}$, where C and δ are constants depending only on α and the geomerty of M and $B_R(q)$. $r(x)$ denotes the geodesic distance from a fixed point $0 \in M$.*

The proof of the above Theorem utilizes the Leray-Schauder degree theory together with many potential theoretic estimates.

§5. Martin Boundary

Our results above rely heavily on the special properties of harmonic functions, and our C^0 estimate technique is the key link from harmonic functions to harmonic maps. Therefore, it is natural to ask what kind of potential theoretic results have corresponding counterpart in harmonic maps. In this section we briefly illustrate how the Dirichlet problem can be posed and solved on the so-called Martin boundary. Let Ω be a complete Riemannian manifold (with or without boundary) which supports a nonconstant positive superharmonic function. For convenience, we denote the interior by Ω. Let $G(x, y)$ be Green's function which vanishes on $\partial\Omega$ and at infinily. Let o be a fixed point. Martin function $h_y(x)$ with pole at y is defined to be $h_y(x)=h(y, x)=\frac{G(y, x)}{G(y, o)}$. Let $\{y_i\}$ be a sequence of points in Ω which has no limit point in the interior of Ω. The corresponding sequence $\{h_{y_i}\}$ of Martin functions is called fundamental, if h_{y_i} converge to a harmonic function, say h_Y. Two fundamental sequences $\{y_i\}$ and $\{z_i\}$ are equivalent if the limiting harmonic functions h_Y and h_Z coincide. The set of all equivalence classes of fundamental sequences is called the Martin boundary of Ω, which is denoted by $\partial^M\Omega$. Or, equivalently, the set of such limiting harmonic functions can be identified with the Martin boundary. Topologized in an obvious way [M], $\bar{\Omega}=\partial^M\Omega\cup\Omega$ is a compact metric space, and Ω is dense in $\bar{\Omega}$. A positive harmonic function h is called minimal, if f is another positive harmonic functions such that $0\leq f\leq h$, then $f=ch$ for some suitable constant c. We define the minimal Martin boundary $\partial_1{}^M\Omega=\{h_Y\in\partial^M\Omega \mid h_Y \text{ is minimal}\}$. In [M], Martin proved that for every positive harmonic function h there exists a measure $d\mu$ concentrated on $\partial_1{}^M\Omega$ such that

$$h(x)=\int_{\partial_1{}^M\Omega} K(x, \xi)\,d\mu(\xi),$$

where $K(x, \xi)$ is the Martin function at $\xi\in\partial_1{}^M\Omega$. Later Brelot showed that the Dirichlet problem can be solved on Martin boundary from a general point of view [B]. It is therefore easy to reformulate our previous results on Dirichlet problem for harmonic maps in this context.

Finally, we would like to mention possible future development from this point of view: There are many other interesting results from harmonic analysis, and it is very interesting to study to what extent the results from the linear theory carry over to nonlinear situation.

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