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ON PSEUDO-UMBILICAL SURFACES IMMERSED IN A EUCLIDEAN m-SPACE E^m

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1. Introduction

In the study of a surface M immersed in a Euclidean m-space E^m, the total mean curvature $\int_M \alpha^2\, dv$ has played a very importment rôle, where α is the lenth of the mean curvature vector and dv is the volume element of M. It is known to be conformal invariant ([3]) and also to be spectral invariant ([9]). Besides, for a compact surface M immersed in E^m the inequality

$$(1.1)\qquad \int_M \alpha^2 dv \geqq 4\pi$$

is proved in [10] and later the inequality has been generalized for any n-dimensional compact submanifold M immersed in E^m by

$$(1.2)\qquad \int_M \alpha^n dv \geqq C_n,$$

where C_n is the volume of the unit n-sphere S^n ([1, I]).

If the compact surface M is specialized as pseudo-umbilical with nonnegative Gauss curvature, then M is more restricted as being homeomorphic to a 2-sphere with an upper bound condition ([5]) indicated by

$$(1.3)\qquad \int_M \alpha^2 dv \leqq (2+\pi)\pi.$$

A pseudo-umbilical surface M immersed in E^m is defined by the

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equation

(1.4) $$\langle h(X, Y),\ H\rangle = \mu\langle X, Y\rangle$$

holding for all tangent vector fields X, Y and for some function μ on M, where h is the second fundamental form and H is the mean curvature vector of M.

We shall study in the paper some properties of a pseudo-umbilical surface M immersed in E^m through the notion of parallel mean curvature vector in its normal bundle by Theorem 1, and prove Theorem 2

(1.5) $$\int_M \alpha^2 dv \geqq \frac{\pi}{C_{m-1}}[\tau(M) + C_{m-1}\chi(M)]$$

for a compact pseudo-umbilical surface immersed in E^m, where $\tau(M)$ is the total absolute curvature and $\chi(M)$ is the Euler characteristic of M. The equality sign of (1.5) holds when and only when $m=3$ or $\lambda_{m-2}=0$ if $m\geqq 4$.

2. Compact surfaces with parallel mean curvature vector in a Euclidean space

Let M be a non-minimal surface in a Euclidean space E^m of dimension m. We choose a local field of orthonormal frames $e_1, e_2, \xi_3, \xi_4, \cdots, \xi_m$ in E^m such that, restricted to M, the vectors e_1, e_2, are tangent to M (and, consequently, the remaining vectors $\xi_3, \xi_4, \cdots, \xi_m$ are normal to M).

We shall make use the following convention on the ranges of indices:

$$1\leqq i, j, k, l, \cdots \leqq 2;\quad 3\leqq r, s, t, \cdots \leqq m;$$

unless otherwise stated.

In terms of canonical forms ω_i and the connection forms ω_{ij}, the structure equations on the surface M are given as follows:

$$d\omega_i = \Sigma\omega_{ij}\wedge\omega_j,\quad \omega_{ij}+\omega_{ji}=0,$$
$$d\omega_{ij} = \Sigma\omega_{ik}\wedge\omega_{kj} + \Omega_{ij},\quad \Omega_{ij} = \frac{1}{2}\Sigma R_{ijkl}\omega_k\wedge\omega_l,$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the curvature tensor) on the surface M. Since $\omega_r=0$, by Cartan's lemma, we may write

(2.1) $$\omega_{ir} = \Sigma h_{ij}{}^r\omega_j,\quad h_{ij}{}^r = h_{ji}{}^r.$$

Thus we have

$$R_{ijkl}=\Sigma(h_{il}{}^r h_{jk}{}^r-h_{ik}{}^r h_{jl}{}^r).$$

Moreover, M being 2-dimensional, the curvature tensor of M can be expressed by

$$R_{ijkl}=k(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}),$$

where k denotes the sectional curvature of the surface M and δ_{ij} is the Kronecker delta. Hence the Ricci tensor R_{ij}, and the scalar curvature R are given by

$$R_{ij}=\frac{1}{2}R\delta_{ij}=\Sigma h^r h_{ij}{}^r-\Sigma h_{il}{}^r h_{jl}{}^r, \tag{2.2}$$

$$R=2k=\Sigma(h^r)^2-\Sigma(h_{ij}{}^r)^2, \tag{2.3}$$

where $h^r=\Sigma h_{ii}{}^r$.

From now on, we shall use the summation convention for the dummy indices.

We take exterior differentiation of (2.1) and define $h_{ijk}{}^r$ by

$$h_{ijk}{}^r\omega_k=dh_{ij}{}^r-h_{ij}{}^r\omega_{lj}-h_{lj}{}^r\omega_{il}+h_{ij}{}^s\omega_{sr}. \tag{2.4}$$

Then $h_{ijk}{}^r$ is the covariant derivative of $h_{ij}{}^r$ (i.e., $\nabla_k h_{ij}{}^r=h_{ijk}{}^r$) and we have $h_{ijk}{}^r=h_{ikj}{}^r$ ([2]).

Suppose that the mean curvature vector is parallel in the normal bundle. Then we can take $h^r=0$ for $r=4, 5, \cdots, m$. And we also obtain

$$R_{ijrs}=h_{il}{}^r h_{jl}{}^s-h_{jl}{}^r h_{il}{}^s. \tag{2.5}$$

Thus

$$R_{ijrs}h^s=(h_{il}{}^r h_{jl}{}^3-h_{jl}{}^r h_{il}{}^3)h^3=0.$$

Since M is a non-minimal surface, $h^3\neq 0$. Therefore

$$h_{il}{}^r h_{jl}{}^3-h_{jl}{}^r h_{il}{}^3=0. \tag{2.6}$$

The covariant derivative of (2.6) gives

$$h_{ilk}{}^r h_{jl}{}^3+h_{il}{}^r h_{jlk}{}^3-h_{jlk}{}^r h_{il}{}^3-h_{jl}{}^r h_{ilk}{}^3=0. \tag{2.7}$$

If we put $R_k=\nabla_k R$, then by (2.2), (2.8) $$\frac{1}{2}R_k\delta_{ij}=h^3h_{ijk}{}^3-h_{ilk}{}^rh_{jl}{}^r-h_{il}{}^rh_{jlk}{}^r.$$

And, if we take skew symmetric part of (2.8) with respect to i and k, then

(2.9) $$\frac{1}{2}(R_k\delta_{ij}-R_i\delta_{jk})=h_{kl}{}^rh_{jli}{}^r-h_{il}{}^rh_{jlk}{}^r.$$

Furthermore, by (2.7), we obtain

$$h_{ilk}{}^rh_{jkm}{}^3h_{jl}{}^3h_{im}{}^r+h_{jlk}{}^3h_{jkm}{}^3h_{il}{}^rh_{im}{}^r-h_{jlk}{}^rh_{jkm}{}^3h_{il}{}^3h_{im}{}^r$$
$$-h_{ilk}{}^3h_{jkm}{}^3h_{jl}{}^rh_{im}{}^r=0.$$

Since, by (2.6) and (2.9),

$$\begin{aligned}h_{jlk}{}^rh_{jkm}{}^3h_{il}{}^3h_{im}{}^r&=h_{jlk}{}^rh_{jkm}{}^3h_{il}{}^rh_{im}{}^3\\&=h_{jkm}{}^3h_{im}{}^3\left[h_{kl}{}^rh_{ilj}{}^r-\frac{1}{2}(R_k\delta_{ij}-R_i\delta_{jk})\right],\end{aligned}$$

$$\begin{aligned}h_{ilk}{}^3h_{jkm}{}^3h_{jl}{}^rh_{im}{}^r-h_{jlk}{}^3h_{jkm}{}^3h_{il}{}^rh_{im}{}^r&\\&=h_{ilk}{}^rh_{jkm}{}^3h_{jl}{}^3h_{im}{}^r-h_{jlk}{}^rh_{jkm}{}^3h_{il}{}^3h_{im}{}^r\\&=\frac{1}{2}h_{jkm}{}^3h_{im}{}^3R_k\delta_{ij}-\frac{1}{2}h_{jkm}{}^3h_{im}{}^3R_i\delta_{jk}\\&=\frac{1}{2}R_kh_{imk}{}^3h_{im}{}^3=\frac{1}{4}R_kh_k,\end{aligned}$$

where $s_3=\sum_{i,j}(h_{ij}{}^3)^2$ and $h_k=\nabla_ks_3$. Hence we have

$$2\sum_{i,j,k,r}\left[\sum_l(h_{ilk}{}^3h_{jl}{}^r-h_{jlk}{}^3h_{il}{}^r)\right]^2=-\sum R_kh_k.$$

Suppose that s_3 is constant. Then we have

$$h_{ilk}{}^3h_{jl}{}^r-h_{jlk}{}^3h_{il}{}^r=0 \text{ for all } i, j, k \text{ and } r.$$

Hence $h_{ilk}{}^3h_{jl}{}^3-h_{jlk}{}^3h_{il}{}^3=0$. Therefore

(2.10) $$h_{ijk}{}^3=0 \text{ for all } i, j \text{ and } k \text{ (see [7])}.$$

On the other hand,

(2.11) $$\begin{aligned}\Delta h_{ij}{}^r&=R_{ik}h_{jk}{}^r-R_{kijl}h_{kl}{}^r\\&=\frac{1}{2}Rh_{ij}{}^r-\frac{1}{2}R(\delta_{kl}\delta_{ij}-\delta_{kj}\delta_{il})h_{kl}{}^r\\&=\frac{1}{2}R(2h_{ij}{}^r-h^r\delta_{ij}),\end{aligned}$$

where Δ denotes the Laplacian on M. Since $\Delta h_{ij}{}^3=0$,

$$(2.12)\qquad R(2h_{ij}{}^3-h^3\delta_{ij})=0.$$

If $R\neq 0$, then $h_{11}{}^3=h_{22}{}^3$ and $h_{12}{}^3=h_{21}{}^3=0$. Hence M is a pseudo-umbilical surface, since $h^r=0$ for $r=4, 5, \cdots, m$.

Thus we have proved the following theorem.

THEOREM 1. *Let M be a non-minimal surface in a Euclidean m-space E^m satisfying*

(1) *mean curvature vector is parallel in the normal bundle, and*

(2) $s_3=\sum_{i,j}(h_{ij}{}^3)^2$ *is constant.*

Then we have the scalar curvature $R=0$ or M is pseudo-umbilical.

COROLLARY. *If M is a compact surface in E^m satisfying the conditions of Theorem 1, then M is a flat or minimal surface in a hypersphere of E^m. In addition, if M has nonnegative scalar curvature, then the second fundamental forms are parallel.*

Proof. Since the Gauss curvature G is given by

$$(2.13)\qquad G=\sum_r[h_{11}{}^r h_{22}{}^r-(h_{12}{}^r)^2]=\frac{1}{2}\sum_{i,j,r}[(h_{11}{}^r+h_{22}{}^r)^2-(h_{ij}{}^r)^2]=\frac{1}{2}R.$$

$G=0$ if $R=0$. Hence M is a flat surface.

Let $R\neq 0$. Then M is a pseudo-umbilical surface such that the mean curvature vector is parallel in the normal bundle.
Therefore M is a minimal surface in a hypersphere of E^m ([12]).

Let $s_r=\sum_{i,j}(h_{ij}{}^r)^2$ for $r=4, 5, \cdots, m$. Then, by (2.11),

$$\begin{aligned}\Delta s_r&=2\sum_{i,j}h_{ij}{}^r\Delta h_{ij}{}^r+2\sum_{i,j,k}(\nabla_k h_{ij}{}^r)^2\\&=2R\sum_{i,j}(h_{ij}{}^r)^2+2\sum_{i,j,k}(h_{ijk}{}^r)^2.\end{aligned}$$

Suppose that M is a compact surface with nonnegative scalar curvature.
Since $\Delta s_r\geqq 0$, by the Stoke's theorem, $\Delta s_r=0$.
Hence $h_{ijk}{}^r=0$ for all i, j, k and $r\geq 4$.
Thus, by (2.10), the second fundamental forms are parallel.

3. Pseudo-umbilical surfaces in a Euclidean space

Let M be a compact pseudo-umbilical surface in E^m. Then we can choose a local field of orthonormal frame $e_1, e_2, \xi_3, \cdots, \xi_m$ defined along M such that e_1, e_2 are tangent, $\xi_3, \xi_4, \cdots, \xi_m$ are normal to M and the Lipschitz-Killing curvature $K(p, \xi)$ is given by

$$(3.1) \qquad K(p, \xi) = \Sigma\lambda_{r-2}(p)\cos^2\theta_r, \quad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{m-2}$$

for a unit normal vector $\xi = \Sigma\cos\theta_r\xi_r$ at p in M, where $\lambda_{r-2} = \det(h_{ij}{}^r)$, $r = 3, 4, \cdots, m$.

Since M is a pseudo-umbilical surface, if we take ξ_3 in the direction of the mean curvature vector, then

$$(3.2) \quad h_{11}{}^3 = h_{22}{}^3 = \alpha, \quad h_{12}{}^3 = h_{21}{}^3 = 0 \text{ and } h^r = 0 \text{ for } r = 4, 5, \cdots, m,$$

where α is the length of the mean curvature vector at p ([5]).

For pseudo-umbilical surfaces, we shall choose a local field of orthonormal frame satisfying the above conditions.

THEOREM 2. *Let M be a compact pseudo-umbilical surface in E^m. Then we have*

$$\int_M \alpha^2 dv \geqq \frac{\pi}{C_{m-1}}[\tau(M) + C_{m-1}\chi(M)],$$

where $\tau(M)$ is the total absolute curvature of M, C_{m-1} is the volume of the unit $(m-1)$-sphere and $\chi(M)$ is the Euler characteristic of M. The equality sign holds when and only when $m=3$ or $\lambda_{m-2}=0$ if $m \geqq 4$.

Proof. Let S_p be the unit $(m-3)$-sphere of all unit normal vectors at p in M and $d\sigma$ the volume element of S_p. Since

$$|K(p, \xi)| = |\Sigma\lambda_{r-2}\cos^2\theta_r| \leqq \lambda_1\cos^2\theta_3 - \sum_{r=4}\lambda_{r-2}\cos^2\theta_r,$$

$$\int_{S_p} |K(p, \xi)|\, d\sigma \leqq \alpha^2 \int_{S_p} \cos^2\theta_3 d\sigma - \sum_{r=4}\lambda_{r-2}\int_{S_p}\cos^2\theta_r d\sigma$$
$$= 2\alpha^2 \frac{C_{m-1}}{2\pi} - \frac{C_{m-1}}{2\pi}G$$

by spherical integration. Hence the total absolute curvature $\tau(M)$ of M is given by

$$\tau(M) \leqq \frac{C_{m-1}}{\pi}\int_M \alpha^2 dv - \frac{C_{m-1}}{2\pi}\int_M G\ dv$$

$$=\frac{C_{m-1}}{\pi}\int_M \alpha^2 dv - C_{m-1}\chi(M).$$

If the equality sign holds and $m\geqq 4$, then $|\Sigma\lambda_{r-2}\cos^2\theta_r|=\lambda_1\cos^2\theta_3-\sum_{r=4}\lambda_{r-2}\cos^2\theta_r$. Thus $\lambda_1=0$ or $\lambda_2=\lambda_3=\cdots=\lambda_{m-2}=0$. Since there does not exist a compact minimal submanifold in a Euclidean space ([11]), $\lambda_1=\alpha^2\neq 0$. Hence

$$\lambda_2=\lambda_3=\cdots=\lambda_{m-2}=0.$$

The converse of this is trivial, since $0\geqq\lambda_2\geqq\cdots\geqq\lambda_{m-2}$.

THEOREM 3. *Let M be a compact pseudo-umbilical surface in E^m with $\lambda_{m-2}=0$. Then M is a 2-sphere of a 3-dimensional linear subspace of E^m.*

Proof. Since $0\geqq\lambda_2\geqq\cdots\geqq\lambda_{m-2}$,

$$\lambda_2=\lambda_3=\cdots=\lambda_{m-2}=0.$$

Hence $G=\lambda_1=\alpha^2$ and the total absolute curvature $\tau(M)$ of M is given by

$$\begin{aligned}\tau(M)&=\int_M\left(\int_{S_p}|K(p,\xi)|\,d\sigma\right)dv\\&=\int_M\lambda_1\left(\int_{S_p}\cos^2\theta_3 d\sigma\right)dv\\&=\frac{C_{m-1}}{2\pi}\int_M G\ dv=C_{m-1}\chi(M)\end{aligned}$$

by the Gauss-Bonnet theorem. Let $\beta(M)$ be the sum of the betti numbers of M. Since $\tau(M)\geqq C_{m-1}\beta(M)$ ([6, I]) and $\chi(M)\leqq\beta(M)$, $\chi(M)=\beta(M)=2$. Therefore

$$\int_M\alpha^2 dv=\int_M G\ dv=2\pi\chi(M)=4\pi.$$

Hence M is an ordinary 2-sphere ([1, I]).

THEOREM 4. *Let M be a compact pseudo-umbilical surface in E^m. Then we have*

$$\sum_{i,j}\sum_{r=4}^{m}\int_M(h_{ij}{}^r)^2 dv\geqq 4\pi(2-\chi(M)).$$

The equality sign holds when and only when M is a 2-sphere of a 3-dimensional linear subspace of E^m, in this case, $h_{ij}{}^r=0$ for all i, j and $r\geqq 4$.

Proof. By (2.3) and (3.2), the scalar curvature R is given by

$$R=4\alpha^2-\Sigma(h_{ij}{}^r)^2=2\alpha^2-\sum_{i,j}\sum_{r=4}^{m}(h_{ij}{}^r)^2.$$

And, by (2.13) and the Gauss-Bonnet theorem, we obtain

$$\int_M R\, dv=4\pi\chi(M).$$

Since $\int_M \alpha^2 dv\geqq 4\pi$,

$$4\pi\chi(M)\geqq 8\pi-\sum_{i,j}\sum_{r=4}^{m}\int_M (h_{ij}{}^r)^2 dv.$$

Hence

$$\sum_{i,j}\sum_{r=4}^{m}\int_M (h_{ij}{}^r)^2 dv\geqq 8\pi-4\pi\chi(M)=4\pi(2-\chi(M)).$$

If the equality sign holds, then $\int_M \alpha^2 dv=4\pi$. Therefore M is an ordinary 2-sphere. In this case, $\sum_{i,j}\sum_{r=4}^{m}\int_M (h_{ij}{}^r)^2 dv=0$, since $\chi(M)=2$. Thus $h_{ij}{}^r=0$ for all i, j and $r\geqq 4$.

The converse of this is trivial.

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