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TUBE FORMULAS IN PRODUCT RIEMANNIAN MANIFOLDS II

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1. Introduction

Let $P\subset M$ denote that P is a compact p-dimensional embedded submanifold of an m-dimensional Riemannian manifold M. Denote by $V_P{}^M(r)$ the m-dimensional volume of the tube $T(P,r)$ of radius r about P and by $A_P{}^M(r)$ the $(m-1)$-dimensional volume of the boundary of $T(P,r)$. It is well-known that for small $r>0$

$$\int_0^r A_P{}^M(r)\,dr = V_P{}^M(r).$$

Throughout this paper we assume that all manifolds are of class C^∞. In [3], the first author proved the following theorem.

THEOREM 1. *Let $P\subset M$ and $Q\subset N$ so that $P\times Q\subset M\times N$, where $\times$ denotes the Riemannian product. Then we have*

$$\tilde{A}^{M\times N}_{P\times Q}(s)=\tilde{A}_P{}^M(s)\tilde{A}_Q{}^N(s), \tag{1}$$

where

$$\tilde{A}_P{}^M(s)=\int_0^\infty e^{-s^2t^2}A_P{}^M(t)\,dt. \tag{2}$$

In this article, applying this theorem and Weyl's tube formula [5], we derive a product formula in Euclidean space. We also give a new derivation of the Nijenhuis formula [4]. To be more specific let

$$V_P{}^M(r)=\frac{(\pi r^2)^{(m-p)/2}}{\Gamma((m-p+2)/2)}\sum_{i=0}^{\infty}\Big(\int_P a_{2i}\,dP\Big)r^{2i} \tag{3}$$

for sufficiently small $r>0$ (see Remark (1)). Here $(\pi r^2)^k/\Gamma((k+2)/2)$

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is the volume of a ball of radius r in $\boldsymbol{R}^k$, and dP is the volume element of P. Similarly expand $V_Q{}^N(r)$ and $V_{P\times Q}^{M\times N}(r)$ as power series in r:

$$V_Q{}^N(r)=\frac{(\pi r^2)^{(n-q)/2}}{\Gamma((n-q+2)/2)}\sum_{j=0}^{\infty}\Big(\int_Q b_{2j}dQ\Big)r^{2j}, \tag{4}$$

$$V_{P\times Q}^{M\times N}(r)=\frac{(\pi r^2)^{(m+n-p-q)/2}}{\Gamma((m+n-p-q+2)/2)}\sum_{k=0}^{\infty}\Big(\int_{P\times Q} c_{2k}d(P\times Q)\Big)r^{2k}. \tag{5}$$

Then we have the following product formulas for coefficients.

THEOREM 2. *For $k\geq 0$, we have*

$$\begin{aligned}&\frac{1}{m+n-p-q}(m+n-p-q)(m+n-p-q+2)\cdots(m+n-p-q+2k)\\ &\times\int_{P\times Q} c_{2k}d(P\times Q)=\sum_{i=0}^{k}\frac{1}{m-p}(m-p)(m-p+2)\cdots(m-p+2i)\\ &\times\frac{1}{n-q}(n-q)(n-q+2)\cdots(n-q+2k-2i)\\ &\times\Big(\int_P a_{2i}dP\Big)\Big(\int_Q(b_{2k-2i}dQ)\Big).\end{aligned} \tag{6}$$

If $P\subset\boldsymbol{R}^m$, then Weyl's tube formula [5] says that

$$\begin{aligned}V_P{}^{R^m}(r)=&\frac{(\pi r^2)^{(m-p)/2}}{\Gamma((m-p+2)/2)}\\ &\times\sum_{c=0}^{[p/2]}\frac{(m-p)k_{2c}(P)r^{2c}}{(m-p)(m-p+2)\cdots(m-p+2c)},\end{aligned} \tag{7}$$

where $k_{2c}(P)$ is the integral over P of the curvature invariant H_{2c} which is given by

$$H_{2c}=\frac{1}{4^c c!}\sum\delta\binom{\alpha}{\beta}R^P_{\alpha_1\alpha_2\beta_1\beta_2}\cdots R^P_{\alpha_{2c-1}\alpha_{2c}\beta_{2c-1}\beta_{2c}}.$$

Here R^P_{abcd} is the component of the curvature tensor R^P of P, $0\leq 2c\leq p$, $\delta\binom{\alpha}{\beta}$ is equal to 1 or -1 according as $\alpha_1,\cdots,\alpha_{2c}$ are distinct and an even or odd permutation of $\beta_1,\cdots,\beta_{2c}$, and otherwise $\delta\binom{\alpha}{\beta}$ is zero. The summation is taken over all α's and β's running from 1 to p.

Combining the formulas (6) and (7) we obtain the following Nijenhuis formula.

THEOREM 3([4]). *Let $P\subset M$ and $Q\subset N$ so that $P\times Q\subset M\times N$. Then we have for $0\leq 2c\leq p+q$*

$$k_{2c}(P\times Q)=\sum_{i=0}^{c}k_{2i}(P)k_{2c-2i}(Q). \tag{8}$$

The Nijenhuis formula (8) and Weyl's formula (7), then, give the following product formula in Euclidean space.

THEOREM 4. *Let $P\subset \boldsymbol{R}^m$ and $Q\subset \boldsymbol{R}^n$ so that $P\times Q\subset \boldsymbol{R}^m\times \boldsymbol{R}^n$. Then we have*

$$A_{P\times Q}^{\boldsymbol{R}^m\times \boldsymbol{R}^n}(r)$$

$$=\sum_{i=0}^{[p/2]}\sum_{j=0}^{[q/2]}\frac{\pi^{(m+n-p-q)/2}k_{2i}(P)k_{2j}(Q)}{2^{i+j-1}\Gamma((m+n-p-q+2i+2j)/2)}r^{(m+n-p-q+2i+2j-1)} \tag{9}$$

and

$$V_{P\times Q}^{\boldsymbol{R}^m\times \boldsymbol{R}^n}(r)$$

$$=\sum_{i=0}^{[p/2]}\sum_{j=0}^{[q/2]}\frac{\pi^{(m+n-p-q)/2}k_{2i}(P)k_{2j}(Q)}{2^{i+j}\Gamma((m+n-p-q+2i+2j+2)/2)}r^{(m+n-p-q+2i+2j)}. \tag{10}$$

Remarks. (1) The first two terms in the power series for $V_P{}^M(r)$ are given by ([2])

$$a_0=1, \text{ and}$$

$$a_2=\frac{1}{2(m-p+2)}\Big(\sum_{a,b=1}^{p}(R^P_{abab}-R^M_{abab})$$

$$-\sum_{a=1}^{p}\sum_{i=p+1}^{m}R^M_{aiai}-\sum_{i,j=p+1}^{m}R^M_{ijij}\Big),$$

where R^P and R^M are the curvature tensors of P and Q respectively.

(2) Since $a_0=b_0=c_0$ in (6), the case $k=0$ implies the trivial relation

volume of $P\times Q$=volume of $P\times$ volume of Q.

(3) Theorem [2] is a generalization of a result by Gray (the formula

(8) in [1, p. 66]).

2. Proofs of Theorems

Proof of Theorem 2. From (2) and (3) we find

$$\tilde{A}_P{}^M(s)=\frac{\pi^{(m-p)/2}}{\Gamma((m-p+2)/2)}\times\sum_i\left(\int_P a_{2i}dP\right)(m-p+2i)\int_0^\infty e^{-s^2r^2}r^{m-p+2i-1}dr.$$

Since $\int_0^\infty t^{2x-1}e^{-t^2}dt=\Gamma(x)/2$ we have

$$\tilde{A}_P{}^M(s)=\frac{\pi^{(m-p)/2}}{\Gamma((m-p+2)/2)}\times\sum_i\Gamma((m-p+2i+2)/2)s^{-(m-p+2i)}\int_P a_{2i}dP. \tag{11}$$

Similarly

$$\tilde{A}_Q{}^N(s)=\frac{\pi^{(n-q)/2}}{\Gamma((n-q+2)/2)}\times\sum_j\Gamma((n-q+2j+2)/2)s^{-(n-q+2j)}\int_Q b_{2j}dQ \tag{12}$$

and

$$\tilde{A}_{P\times Q}^{M\times N}(s)=\frac{\pi^{(m+n-p-q)/2}}{\Gamma((m+n-p-q+2)/2)}\sum_k\Gamma((m+n-p-q+2k+2)/2)\times s^{-(m+n-p-q+2k)}\int_{P\times Q}c_{2k}d(P\times Q). \tag{13}$$

Therefore from (1) we obtain

$$\sum_k\frac{1}{(m+n-p-q)/2}\left(\frac{m+n-p-q}{2}\right)\left(\frac{m+n-p-q}{2}+1\right)\cdots\left(\frac{m+n-p-q}{2}+k\right)s^{-(m+n-p-q+2k)}\int_{P\times Q}c_{2k}d(P\times Q)$$
$$=\{\sum_i\frac{1}{(m-p)/2}\left(\frac{m-p}{2}\right)\left(\frac{m-p}{2}+1\right)\cdots\left(\frac{m-p}{2}+i\right)\times s^{-(m-p+2i)}\int_P a_{2i}dP\}\ \{\sum_j\frac{1}{(n-q)/2}\left(\frac{n-q}{2}\right)\left(\frac{n-q}{2}+1\right)$$

$$\cdots\left(\frac{n-q}{2}+j\right)s^{-(n-q+2j)}\int_Q b_{2j}dQ\}.$$

By equating powers of s we get (6).

Proof of Theorem 3. From (3) and Weyl's tube formula (7) it is easy to see that

$$\int_P a_{2i}dP=\frac{(m-p)k_{2i}(P)}{(m-p)(m-p+2)\cdots(m-p+2i)}.$$

Similarly we also have

$$\int_Q b_{2j}dQ=\frac{(n-q)k_{2j}(Q)}{(n-q)(n-q+2)\cdots(n-q+2j)}$$

and

$$\int_{P\times Q}c_{2k}d(P\times Q)$$
$$=\frac{(m+n-p-q)k_{2k}(P\times Q)}{(m+n-p-q)(m+n-p-q+2)\cdots(m+n-p-q+2k)}.$$

Putting these equations to (6) we obtain (8).

Proof of Theorem 4. It suffices to prove (10) since (9) is obtained by differentiating (10) with respect to r. According to $P\times Q\subset \boldsymbol{R}^m\times \boldsymbol{R}^n=\boldsymbol{R}^{m+n}$, we have from (7)

$$V_{P\times Q}^{R^m\times R^n}(r)=\sum_{c=0}^{[(p+q)/2]}\frac{\pi^{(m+n-p-q)/2}k_{2c}(P\times Q)}{2^c\Gamma((m+n-p-q+2C+2)/2)}r^{m+n-p-q+2c}.$$

Now by the Nijenhuis formula (8), we get (10).

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