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SOME PROPERTIES OF THE FEYNMAN INTEGRAL

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1. Priliminaries and Notations

In 1976, Cameron and Storvick introduced an operator-valued Yeh-Wiener integral[2]. Using the Neumann series, they calculated an operator-valued Yeh-Wiener integral for some functionals and they found that this integral is a solution of a Wiener integral equation. In this paper, we will calculate an operator-valued Yeh-Wiener integral for a larger class of functionals which contains the functionals in [2]. The method we use here is quite different from the method used in [2]. Moreover, we find that it satisfies the Wiener integral equation [2].

Let $R=\{(s,t)\,|\,a\leqq s\leqq t,\ \alpha\leqq t\leqq\beta\}$ and let $C_1[\alpha,\beta]=\{\eta(\cdot)\,|\,\eta$ is continuous on $[\alpha,\beta]$ and $\eta(\alpha)=0\}$. Let $C_2[R]=\{x(\cdot,\cdot)\,|\,x$ is continuous on R and $x(a,\cdot)=x(\cdot,\alpha)=0\}$ and let $C^*[R]=\{x(\cdot,\cdot)\,|\,x$ is continuous on R and $x(\cdot,\alpha)=0\}$. Let m_1 be the complete Wiener measure on $C_1[\alpha,\beta]$ and let m_2 be the complete Yeh-Wiener measure on $C_2[R]$ [see 6]. For $p>0$ and a scale invariant measurable subset B of $C_1[\alpha,\beta]$, we define a measure $m_1{}^p$ given by $m_1{}^p(B)=m_1(p^{-1}B)$ [see 5]. Let $W(C_1[\alpha,\beta])$ be a class of Wiener measurable functionals ψ defined on $C_1[\alpha,\beta]$ such that $\psi(\gamma y(\cdot)+\eta(\cdot))$ is Wiener integrable in y over $C_1[\alpha,\beta]$ for each positive γ and each η in $C_1[\alpha,\beta]$. The operator-valued Yeh-Wiener integral $I_{\lambda,a}(F)\equiv I_\lambda(F)$ is defined so as to take a functional ψ into the functional $I_\lambda(F)\psi$ whose value at η is

$$[I_\lambda(F)\psi]\eta(\cdot)=\int_{C_2[R]}F(\lambda^{-1/2}x(\cdot,\cdot)+\eta(\cdot))\times\psi(\lambda^{-1/2}x(b,\cdot)+\eta(\cdot))dm_2(x).$$

Here the independent variables for η is the second independent variables for x and F is a functional on $C_2{}^*[R]$.

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Throughout this paper, let θ be a Lebesgue$\times$Wiener measurable functional defined on $[a, b]\times C_1[\alpha, \beta]$ such that θ is defined on everywhere and

$$\|\theta\|_{\infty 1}=\int_{[a,b]}\|\theta(s, \ )\|_{s-\infty}\, dm_L(s)<\infty$$

where m_L is the Lebesgue measure on $[a, b]$ and $\|\theta(s, \cdot)\|_{s-\infty}=\sup\{M|$ the set of η in $C_1[\alpha, \beta]$ with $|\theta(s, \eta)|>M$ is a s-null set$\}$ [see 5]. And let F be a functional on $C_2^*[R]$ given by

$$F(x(\cdot, \cdot))=\int_{[a,b]}\theta(s, x(s, \cdot))dm_L(s).$$

We finish this section with the following lemma.

LEMMA. *Under the above assumptions, for $\lambda>0$, $F(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))$ is difined and satisfies*

$$|F(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))|\leq\|\theta\|_{\infty 1}$$

for $m_2\times m_1$-a.e. (x, η) in $C_2[R]\times C_1[\alpha, \beta]$.

Proof. Let $H_\lambda : [a, b]\times C_2[R]\times C_1[\alpha, \beta]\rightarrow[a, b]\times C_1[\alpha, \beta]$ be a function with $H_\lambda(s, x, \eta)=(s, \lambda^{-1/2}x(s, \cdot)+\eta(\cdot))$. H_λ is everywhere defined and continuous and so $\theta\circ H$ is certainly measurable. Let

$$N=\{(s, \eta)\in[a, b]\times C_1[\alpha, \beta]\,|\,|\theta(s, \eta)|>\|\theta(s, \ )\|_{s-\infty}\}.$$

By the Fubini theorem, N is a null set. And $N^{(s)}$ is a s-null set for s in $[a, b]$ where $N^{(s)}$ is a section of N for s. Also

$$\begin{aligned}&[H_\lambda^{-1}(N)]^{(s,\eta)}\\&=\{x\in C_2[R]\,|\,x(s, \cdot)\in\lambda^{-1/2}[N^{(s)}-\eta(\cdot)]\}.\end{aligned}$$

By Corollary 15 in [5, p. 164], $N^{(s)}-\eta(\cdot)$ is $m_1{}^p$-null except of at most a s-null set of η's where $p=\{\lambda(s-\alpha)/2\}^{-1/2}$. Then $\lambda^{-1/2}[N^{(s)}-\eta(\cdot)]$ is a $m_1{}^q$-null set except of at most a s-null set of η's where $q=\{(s-\alpha)/2\}^{-1/2}$. From Theorem 1 in [1, p. 20], $[H_\lambda^{-1}(N)]^{(s,\eta)}$ is a m_2-null set except of at most a s-null set of η's. Hence, by the Fubini theorem, $H_\lambda^{-1}(N)$ is $m_L\times m_2\times m_1$-null. Therefore, for $m_2\times m_1$-a.e. (x, η) and for m_L-a.e. s,

$$|\theta(s, \lambda^{-1/2}x(s, \cdot)+\eta(\cdot))|<\|\theta(s, \cdot)\|_{s-\infty}.$$

Hence, for $m_2\times m_1$-a.e. (x, η),

$$\begin{aligned}&\int_{[a,b]}|\theta(s, \lambda^{-1/2}x(s, \cdot)+\eta(\cdot))|dm_L(s)\\ &\leqq\int_{[a,b]}\|\theta(s, \cdot)\|_{s-\infty}\,dm_L(s)\\ &=\|\theta\|_{\infty 1}\\ &<\infty.\end{aligned}$$

Thus $F(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))$ is defined and $|(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))|\leqq\|\theta\|_{s-\infty}$ for $m_2\times m_1$-a.e. (x, η) in $C_2[R]\times C_1[\alpha, \beta]$. The proof of this lemma is complete.

2. An operator-valued Yeh-Wiener integral and a Wiener integral equation

First of all, we calculate an operator-valued Yeh-Wiener integral for $[F(x(\cdot, \cdot))]^n$. Let

$$\Delta_{n,a}=\{(s_1, s_2, \cdots, s_n)\,|\,a<s_1<s_2<\cdots<s_n<b\} \text{ and } s_0=a, \quad s_{n+1}=b.$$

THEOREM 1. *Suppose* ψ *is in* $W(C_1[\alpha, \beta])$. *Then for* m_1*-a.e.* η *in* $C_1[\alpha, \beta]$ *and* $\lambda>0$,

$$[I_\lambda(F^n)\psi]\eta(\cdot)=n!\int_{\Delta_{n,a}}\int_{\prod_{k=1}^{n+1}C_1[\alpha,\beta]}\{\prod_{k=1}^{n}\theta(s_k, \lambda^{-1/2}\sum_{j=1}^{k}p_jw_j(\cdot)+\eta(\cdot))\}\psi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_jw_j(\cdot)+\eta(\cdot))\,d\prod_{j=1}^{n+1}m_1(w_j)\,d\prod_{k=1}^{n+1}m_L(s_k)$$

where $p_j=\{(s_j-s_{j-1})/2\}^{-1/2}$ *for* $j=1, 2, \cdots, n+1$.

Proof. From Lemma 1, there is a null subset N of $C_2[R]\times C_1[\alpha, \beta]$ such that $F(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))$ is defined on N^c. By the Fubini theorem, there is a null subset N_1 of $C_1[\alpha, \beta]$ such that for η in N_1^c, $N^{(\eta)}$ is m_2-null. Let η be given in N_1^c and $\lambda>0$ be given. Then

$$\begin{aligned}&[I_\lambda(F^n)\psi]\eta(\cdot)\\ &=\int_{C_2[R]}\{\int_{[a,b]}\theta(s, \lambda^{-1/2}x(s, \cdot)+\eta(\cdot))dm_L(s)\}^n\\ &\quad\times\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))dm_2(x)\end{aligned}$$

$$\overset{(1)}{=} \int_{C_2[R]} \int_{\prod_{k=1}^{n}[a,b]} \{\prod_{k=1}^{n} \theta(s_k, \lambda^{-1/2}x(s_k, \cdot)+\eta(\cdot))\} \times\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))dm_2(x)$$

$$\overset{(2)}{=} n! \int_{C_2[R]} \int_{\Delta_{n,a}} \{\prod_{k=1}^{n} \theta(s_k, \lambda^{-1/2}x(s_k, \cdot)+\eta(\cdot))\} \times\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))d\prod_{k=1}^{n} m_L(s_k)dm_2(x)$$

$$\overset{(3)}{=} n! \int_{\Delta_{n,a}} \int_{C_2[R]} \{\prod_{k=1}^{n} \theta(s_k, \lambda^{-1/2}x(s_k, \cdot)+\eta(\cdot))\} \times\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))dm_2(x)d\prod_{k=1}^{n} m_L(s_k)$$

$$\overset{(4)}{=} n! \int_{\Delta_{n,a}} \int_{\prod_{k=1}^{n+1} C_1[\alpha,\beta]} \{\prod_{k=1}^{n} \theta(s_k, \lambda^{-1/2}\sum_{j=1}^{k} p_j w_j(\cdot)+\eta(\cdot))\} \times\psi(\lambda^{-1/2}\sum_{j=1}^{n+1} p_j w_j(\cdot)+\eta(\cdot))d\prod_{j=1}^{n+1} m_1(w_j)d\prod_{k=1}^{n+1} m_L(s_k).$$

Step (1) follows from the Fubini theorem. Since the integrand is invariant under permutations of s-variables, the integral over the $n!$ simplexes are equal. Hence we obtain Step (2). Step (3) follows from the Fubini theorem which will be justified below. Step (4) follows from the n-parallel lines theorem [1, p. 23].

Now,

$$\int_{C_2[R]} |F(\lambda^{-1/2}x(\cdot, \cdot)+\eta(\cdot))| \, |\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))| dm_2(x)$$

$$\overset{[1]}{\leqq} \|\theta\|_{\infty 1}^{n} \int_{C_2[R]} |\psi(\lambda^{-1/2}x(b, \cdot)+\eta(\cdot))| dm_2(x)$$

$$\overset{[2]}{=} \|\theta\|_{\infty 1}^{n} \int_{C_1[\alpha,\beta]} |\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}y(\cdot)+\eta(\cdot))| dm_1(y)$$

$$\overset{[3]}{<} \infty.$$

Step [1] results from Lemma in section 1. Step [2] results from the one-line theorem [1, p. 21]. From the definition of $W(C_1[\alpha, \beta])$, we have Step [3].

This justifies the use of the Fubini theorem in Step (3) above. Thus the proof of this theorem is complete.

Applying the dominated convergence theorem, we obtain the following corollary.

COROLLARY. *Let* $F(x(\cdot,\cdot))=\exp\{\int_{[a,b]}\theta(s,\lambda^{-1/2}x(b,\cdot))dm_L(s)\}$ *for* x *in* $C_2^*[R]$. *Then for* m_1-*a.e.* η *and* $\lambda>0$,

$$[I_\lambda(F)\psi](\cdot)=\sum_{n=0}^{\infty}\int_{\Delta_{n,a}}\int_{\prod_{k=1}^{n+1}C_1[\alpha,\beta]}\{\prod_{k=1}^{n}\theta(s_k,\lambda^{-1/2}\sum_{j=1}^{k}p_jw_j(\cdot)$$

$$+\eta(\cdot))\}\psi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_jw_j(\cdot)+\eta(\cdot))d\prod_{j=1}^{n+1}m_1(w_j)d\prod_{k=1}^{n}m_L(s_k).$$

This value is denoted by $G(a,\eta)$.

THEOREM 2. *Under the assumptions in Lemma in section* 1 *and Theorem* 1, G *satisfies the Wiener integral equation for a.e.*-η;

$$G(a,\eta)=\int_{C_1[\alpha,\beta]}\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)$$
$$+\int_{[a,b]}\int_{C_1[\alpha,\beta]}\theta(s,\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))$$
$$\times G(s,\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)dm_L(s)$$

for a.e.-η.

Proof. From the following equalities, the proof of this theorem is complete.

$$\int_{C_1[\alpha,\beta]}\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)$$
$$+\int_{[a,b]}\int_{C_1[\alpha,\beta]}\theta(s,\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))$$
$$\times G(s,\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)dm_L(s)$$
$$\overset{(1)}{=}\int_{C_1[\alpha,\beta]}\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)$$
$$+\sum_{n=0}^{\infty}\int_{[a,b]}\int_{C_1[\alpha,\beta]}\psi(s,\lambda^{-1/2}\{(s-a)/2\}^{1/2}w(\cdot)$$
$$+\eta(\cdot))[\int_{\Delta_{n,s}}\int_{\prod_{k=1}^{n+1}C_1[\alpha,\beta]}\{\prod_{k=1}^{n}\theta(s_k,\lambda^{-1/2}\sum_{j=1}^{k}p_jw_j(\cdot)$$

$$+\lambda^{-1/2}\{(s-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))\}\psi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_jw_j(\cdot)$$

$$+\lambda^{-1/2}\{(s-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))\ d\prod_{j=1}^{n+1}m_1(w_j)d\prod_{k=1}^{n}m_L(s_k)]$$

$$\times dm_1(w)dm_L(s)$$

$$\overset{(2)}{=}\int_{C_1[\alpha,\beta]}\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)$$

$$+\sum_{n=0}^{\infty}\int_{\Delta_{n+1,a}}\int_{\prod_{k=1}^{n+2}C_1[\alpha,\beta]}\{\prod_{k=1}^{n+1}\theta(s_k,\lambda^{-1/1}\sum_{j=1}^{k}p_jw_j(\cdot)+\eta(\cdot))\}$$

$$\times\psi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_jw_j(\cdot)+\eta(\cdot))d\prod_{j=1}^{n+2}m_1(w_j)d\prod_{k=1}^{n+1}m_L(s_k)$$

$$\overset{(3)}{=}\int_{C_1[\alpha,\beta]}\psi(\lambda^{-1/2}\{(b-a)/2\}^{1/2}w(\cdot)+\eta(\cdot))dm_1(w)$$

$$+\sum_{n=1}^{\infty}\int_{\Delta_{n,a}}\int_{\prod_{k=1}^{n+1}C_1[\alpha,\beta]}\{\prod_{k=1}^{n}\theta(s_k,\lambda^{-1/2}\sum_{j=k}^{k}p_jw_j(\cdot)+\eta(\cdot))\}$$

$$\times\psi(\lambda^{-1/2}\sum_{j=1}^{n}p_jw_j(\cdot)+\eta(\cdot))d\prod_{j=1}^{n+1}m_1(w_j)d\prod_{k=1}^{n}m_L(s_k)$$

$$\overset{(4)}{=}G(a,\eta).$$

From Corollary of Theorem 1, Step (1) holds for a.e.-η. By the dominated convergence theorem, we obtain Step (2). Let $s_1=s$ and $s_k=s_{k-1}$ for $k>2$. Then we have Step (3). Step (4) follows from the denfiition of G.

EXAMPLE. Let Φ be a measurable function everywhere defined on $[a,b]\times[\alpha,\beta]\times C_1[\alpha,\beta]$ such that

$$\int_{[a,b]}\int_{[\alpha,\beta]}\|\Phi(s,t,\cdot)\|_{s-\infty}dm_L(t)dm_L(s)<\infty.$$

Let F be a functional on $C_2{}^*[R]$ given by

$$F(y)=\{\int_{[a,b]}\int_{[\alpha,\beta]}\Phi(s,t,y(s,\cdot))dm_L(t)dm_L(s)\}^n.$$

Suppose ψ is in $W(C_1[\alpha,\beta])$ such that $\psi(x)=\phi(x(\beta))$ for some ϕ in

$L_\infty(R)$. Let θ be a functional on $[a,b]\times C_1[\alpha,\beta]$ given by $\theta(s,\eta)=\int_{[\alpha,\beta]}\Phi(s,t,\eta(t))dm_L(t)$, $\|\theta\|_{\infty 1}<\infty$. Hence, by Lemma in section 1, for $\lambda>0$, $F(\lambda^{-1/2}x(\cdot,\cdot)+\eta(\cdot))$ is defined for $m_2\times m_1$-a.e. (x,η) in $C_2[R]\times C_1[\alpha,\beta]$. From Theorem 1, $[I_\lambda(F^n)\psi]\eta(\cdot)$ exists for a.e.-η in $C_1[\alpha,\beta]$. Let $\Delta_{n,\alpha}=\{(s_1,s_2,\cdots,s_n)\,|\,\alpha=s_0<s_1<s_2<\cdots<s_n<\beta\}$. By Theorem 1, the Wiener integration formula[6] and the change of parameter, the following equalites hold for a.e.-η;

$$[I_\lambda(F^n)\psi]\eta(\cdot)$$

$$=n!\int_{\Delta_{n,\alpha}}\int_{\prod_{k=1}^{n+1}C_1[\alpha,\beta]}\{\prod_{k=1}^{n}\theta(s_k,\lambda^{-1/2}\sum_{j=1}^{k}p_jw_j(\cdot)+\eta(\cdot))\}$$

$$\times\psi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_jw_j(\cdot)+\eta(\cdot))d\prod_{j=1}^{n+1}m_1(w_j)d\prod_{k=1}^{n}m_L(s_k)$$

$$=(n!)^2\int_{\Delta_{n,\alpha}}\int_{\Delta_{n,\alpha}}\int_{-\infty}^{\infty}\cdots\cdots\int_{-\infty}^{\infty}\{\prod_{k=1}^{n}\Phi(s_k,t_k,\lambda^{-1/2}\sum_{j=1}^{k}p_ju_{j,k}+\eta(t_k))\}$$

$$\times\phi(\lambda^{-1/2}\sum_{j=1}^{n+1}p_ju_{j,n+1}+\eta(\beta))[\prod_{k=1}^{n+1}\prod_{j=k}^{n+1}\{2\pi(t_k-t_{k-1})\}^{-1/2}$$

$$\times\exp\{-(u_{j,k}-u_{j,k-1})^2/2(t_k-t_{k-1})\}]$$

$$\times d\prod_{k=1}^{n+1}\prod_{j=k}^{n+1}m_L(u_{j,k})d\prod_{k=1}^{n}m_L(t_k)d\prod_{k=1}^{n}m_L(s_k)$$

$$=(n!)^2\lambda^{(n+1)(n+2)/4}\int_{\Delta_{n,\alpha}}\int_{\Delta_{n,\alpha}}\int_{-\infty}^{\infty}\cdots\cdots\int_{-\infty}^{\infty}\{\prod_{k=1}^{n}\Phi(s_k,t_k,\sum_{j=1}^{n}v_{j,k}$$

$$+\eta(t_k))\}\phi(\sum_{j=1}^{n}v_{j,n+1}+\eta(\beta))\prod_{k=1}^{n+1}\prod_{j=k}^{n+1}[p_j^{-1}\{2\pi(t_k-t_{k-1})\}^{-1/2}$$

$$\times\exp\{-\lambda(v_{j,k}-v_{j,k-1})^2/(2p_j^2(t_k-t_{k-1}))\}]$$

$$\times d\prod_{k=1}^{n+1}\prod_{j=k}^{n+1}m_L(v_{j,k})d\prod_{k=1}^{n}m_L(t_k)d\prod_{k=1}^{n}m_L(s_k).$$

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