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PROBABILISTIC PROPERTIES ABOUT PERMANENT OF RANDOM (0, 1)-MATRICES

WI CHONG AHN, BONG DAE CHOI AND KYUNG HYUNE LEE

Let $C(n, m, N)$ be the set of all $n\times m$ Boolean matrices among the components of which there are exactly N components are equal to 1, all the other components are equal to 0. The total number of matrices $\omega\in C(n, m, N)$ is given by $\binom{nm}{N}$. Let P be a uniform distribution on $C(n, m, N)$, i.e., $P(\omega)=\binom{nm}{N}^{-1}$. In other words, each of the $\binom{nm}{N}$ elements of the set $C(n, m, N)$ has the same probability $\binom{nm}{N}^{-1}$ to be selected.

The permanent of $n\times m$ matrix $\omega=(\omega_{ij})$, written by $\operatorname{per}(\omega)$, is defined by

$$\operatorname{per}(\omega)=\sum_{\sigma}\omega_{1\sigma(1)}\omega_{2\sigma(1)}\cdots\omega_{n\sigma(n)}$$

where the summation extends over all $1-1$ maps from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, m\}$, $(n\leq m)$. If $n=m$, then the terms in $\operatorname{per}(\omega)$ are, apart from the sign, just terms in the expansion of $\det(\omega)$.

P. Erdos and A. Renyi[3, 4] have investigated the limiting probability of the event that random matrix ω has a positive permanent when $n=m$. This paper is concerned with the limiting probability of the event as above in the case of $n\leq m$.

A row or a column all elements of which are equal to 0 is called a 0-row or a 0-column for the sake of brevity.

Let

$$R(n, m, N)=\{\omega\in C(n, m, N)\,|\,\omega \text{ does not have 0-rows}\}.$$

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LEMMA 1. $|R(n,m,N)|=\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\binom{(n-i)m}{N}$.

If $N=\frac{nm}{2}$, *then* $\frac{|R(n,m,N)|}{|C(n,m,N)|}\to 1$ *as* $n\leq m\to\infty$.

Proof. Let P be the uniform distribution on $C(n,m,N)$. Define a random variable X on $C(n,m,N)$ by

$$X(\omega)=\text{the number of 0-rows of matrix } \omega.$$

Then X can be written as

$$X=\sum_{i=1}^{n}\varepsilon_i$$

where

$$\varepsilon_i=\begin{cases}1 & \text{if } i\text{-th row is all of zeros,}\\ 0 & \text{otherwise}\end{cases}$$

Let $B_k=E\binom{X}{k}$ be the binomial moments of X. It is known [5] that

$$P(X=j)=\sum_{k=j}^{n}(-1)^{k-j}\binom{k}{j}B_k.$$

Now let us show how to calculate B_k,

$$\begin{aligned}B_k=E\binom{X}{k}&=E\Big(\sum_{\substack{k_1+\cdots+k_n=k\\ k_i=0,1}}\binom{\varepsilon_1}{k_1}\binom{\varepsilon_2}{k_2}\cdots\binom{\varepsilon_n}{k_n}\Big)\\
&=E\Big(\sum_{j_1<\cdots<j_k}\varepsilon_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_k}\Big)\\
&=\sum_{j_1<j_2<\cdots<j_k}E(\varepsilon_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_k})\\
&=\sum_{j_1<j_2<\cdots<j_k}P(\varepsilon_{j_1}=1,\ \varepsilon_{j_2}=1,\ \cdots,\ \varepsilon_{j_k}=1)\\
&=\binom{n}{k}\frac{|C(n-k,m,N)|}{|C(n,m,N)|}.\end{aligned}$$

Since $P(X=0)$ is the probability of the event of matrices without 0-rows, we have

$$P(X=0)=\frac{|R(n,m,N)|}{|C(n,m,N)|}.$$

Thus we have

$$|R(n,m,N)|=|C(n,m,N)|P(X=0)$$
$$=|C(n,m,N)|\sum_{j=0}^{n}(-1)^{j}B_j$$
$$=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{(n-j)m}{N}.$$

To show that $\frac{|R(n,m,N)|}{|C(n,m,N)|}\to 1$ as $m\geq n\to\infty$.

We calculate

$$\frac{|R(n,m,N)|}{|C(n,m,N)|}=\frac{\sum_{i=0}^{n-1}(-1)^{i}\binom{n}{i}\binom{(n-i)m}{N}}{\binom{nm}{N}}$$
$$\geq\frac{\binom{nm}{N}-n\binom{(n-1)m}{N}}{\binom{nm}{N}}=1-\frac{n\binom{(n-1)m}{N}}{\binom{nm}{N}}$$
$$\sim 1-n\left(\frac{(n-1)m}{nm}\right)^{N}=1-n\left(1-\frac{1}{n}\right)^{nm/2}$$
$$\sim 1-n\exp\left(-\frac{m}{2}\right)=1+o(1).$$

Here we used the Bonferroni's inequality[5] and

$$\binom{nm}{N}\sim\frac{(nm)^{N}}{N!}.$$

LEMMA 2. *Let $\gamma_{n,m}$ be the number of matrices in $R(n,m,N)$ without 0-columns. Then*

$$\gamma_{n,m}=\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}|R(n,m-j,N)|.$$

Proof. The proof is similar to that of lemma 1 and is omitted.

LEMMA 3. *If $N=\frac{nm}{2}$ and $m(n)=o(\exp n/2)$, then*

$$\frac{\gamma_{n,m}}{|R(n,m,N)|}\to 1 \text{ as } n\leq m\to\infty.$$

Proof. By lemma 2 and Bonferroni's inequality, we have

$$\frac{\gamma_{n,m}}{|R(n,m,N)|}\geq 1-m\frac{|R(n,m-1,N)|}{|R(n,m,N)|}.$$

By lemma 1, we have

$$m\frac{|R(n,m-1,N)|}{|R(n,m,N)|}$$

$$\leq m\frac{\binom{(m-1)n}{N}-n\binom{(m-1)(n-1)}{N}+\binom{n}{2}\binom{(n-2)(m-1)}{N}}{\binom{nm}{N}-n\binom{m(n-1)}{N}}$$

$$\sim m\frac{\left(\frac{n(m-1)}{nm}\right)^N-n\left(\frac{(n-1)(m-1)}{nm}\right)^N+\binom{n}{2}\left(\frac{(n-2)(m-1)}{nm}\right)^N}{1-n\left(\frac{(n-1)m}{nm}\right)^N}$$

$$=m\exp\left(-\frac{n}{2}\right)\left(1+\frac{\frac{n(n-1)}{2}\exp(-m)}{1-n\exp\left(-\frac{m}{2}\right)}\right)$$

$$=o(1).$$

Thus lemma is proved.

LEMMA 4. *Let $\delta_{n,m}$ be the number of matrices in $R(n,m,N)$ without 0-columns and* $\mathrm{per}(\omega)=0$. *Then*

$$\delta_{n,m}\leq 2\binom{2m}{m+1}\binom{nm-2(m-1)}{N}\quad \textit{for } m\geq 5.$$

If $N=\frac{nm}{2}$, *then* $\frac{\delta_{nm}}{|R(n,m,N)|}\to 0$ *as* $m\geq n\to\infty$.

Proof. By the König-Frobenius theorem, we have

$$\delta_{n,m}\leq\sum_{k=2}^{n-1}\binom{n}{k}\binom{m}{m-k+1}\binom{nm-k(m-k+1)}{N}$$

$$\leq\sum_{k=2}^{m-1}\binom{m}{k}\binom{m}{m-k+1}\binom{nm-k(m-k+1)}{N}$$

$$=\binom{m}{2}\binom{m}{m-1}\binom{nm-2(m-1)}{N}+\binom{m}{m-1}\binom{m}{2}\binom{nm-2(m-1)}{N}$$

$$+\sum_{k=3}^{m-2}\binom{m}{k}\binom{m}{m-k+1}\binom{nm-k(m-k+1)}{N}.$$

Since $nm-3(m-2)\geq nm-k(m-k+1)$ for all k with $3\leq k\leq m-2$ and $m\geq 5$, we have

$$\delta_{n,m}\leq m^3\binom{nm-2(m-1)}{N}+\binom{nm-3(m-2)}{N}\sum_{k=3}^{m-2}\binom{m}{k}\binom{m}{m-k+1}$$

$$\leq m^3\binom{nm-2(m-1)}{N}+\binom{nm-3(m-2)}{N}\sum_{k=1}^{m}\binom{m}{k}\binom{m}{m-k+1}$$

$$\leq m^3\binom{nm-2(m-1)}{N}+\binom{nm-2(m-1)}{N}\binom{2m}{m+1}.$$

The last inequality is obtained from the facts that $nm-2(m-1)\geq nm-3(m-2)$ for all $m\geq 4$ and

$$\sum_{k=1}^{m}\binom{m}{k}\binom{m}{m-k+1}=\binom{2m}{m+1}.$$

Since $m^3\leq\binom{2m}{m+1}$ for $m\geq 5$, we have

$$\delta_{n,m}\leq 2\binom{2m}{m+1}\binom{nm-2(m-1)}{N}\quad\text{for } m\geq 5.$$

Next we prove that

$$\frac{\delta_{n,m}}{|R(n,m,N)|}\to 0 \text{ as } m\geq n\to\infty$$

Since

$$\frac{\delta_{n,m}}{|R(n,m,N)|}\leq\frac{2\binom{2m}{m+1}\binom{nm-2(m-1)}{N}}{\binom{nm}{N}-n\binom{(n-1)m}{N}},$$

it is enough to show that

$$\frac{\binom{nm}{N}-\binom{(n-1)m}{N}}{\binom{2m}{m+1}\binom{nm-2(m-1)}{N}}\to\infty.$$

It can be shown by stirling's formula that

(a) $$\frac{\binom{nm}{N}}{\binom{nm-2(m-1)}{N}} \sim 4^{(m-1)}$$

(b) $$\frac{1}{\binom{2m}{m+1}} \sim \frac{\sqrt{m}}{4^m}$$

(c) $$\frac{n\binom{(n-1)m}{N}}{\binom{nm-2(m-1)}{N}} \sim n2^m$$

Hence by (a) and (b), we have

$$\frac{\binom{nm}{N}}{\binom{2m}{m+1}\binom{nm-2(m-1)}{N}} \sim \frac{\sqrt{m}}{4} \to \infty \text{ as } m \geq n \to \infty.$$

By (b) and (c), we have

$$\frac{n\binom{(n-1)m}{N}}{\binom{2m}{m+1}\binom{nm-2(m-1)}{N}} \sim \frac{n\sqrt{m}}{2^m} \to 0 \text{ as } m \geq n \to \infty.$$

Thus lemma is proved.

THEOREM 5. *Let P be the uniform distribution on $R(n,m,N)$. If $N=\frac{nm}{2}$ and $m=o(\exp n/2)$, then*

$$P\{\omega \in R(n,m,N) : \text{per}(\omega) > 0\} \to 1.$$

Proof. Let $C_{nm}^{(j)} = \{\omega \in R(n,m,N) : \omega$ has exactly j 0-columns$\}$ and $A_{nm} = \{\omega \in R(n,m,N) : \text{per}(\omega) = 0\}$.

Then

$$A_{nm} = \bigcup_{j=0}^{m-1} (A_{nm} \cap C_{nm}^{(j)}).$$

Hence

$$P(A_{nm})=\sum_{j=0}^{m-1}P(A_{nm}\cap C_{nm}^{(j)})$$

$$=\sum_{j=0}^{m-n}P(A_{nm}\cap C_{nm}^{(j)})+\sum_{j=m-n+1}^{m-1}P(A_{nm}\cap C_{nm}^{(j)}).$$

For $m-n+1\leq j\leq m-1$, it follows from König-Frobenius theorem that we have $C_{nm}^{(j)}\subset A_{nm}$.

Thus $P(A_{nm}\cap C_{nm}^{(j)})=P(C_{nm}^{(j)})$ for $m-n+1\leq j\leq m-1$. It can be written that

$$P(C_{nm}^{(j)})=\frac{\binom{m}{j}\gamma_{n,m-j}}{|R(n,m,N)|},$$

where $\gamma_{n,r}$ is the number of all Boolean ω's in $R(n,r,N)$ which do not have 0-columns. Note that

$$\sum_{j=0}^{m-1}\binom{m}{j}\gamma_{n,m-j}=|R(n,m,N)|.$$

For $0\leq j\leq m-n$, we have

$$P(A_{nm}\cap C_{nm}^{(j)})=\frac{\binom{m}{j}\delta_{n,m-j}}{|R(n,m,N)|}$$

where $\delta_{n,r}$ is the number of all Boolean ω's in $R(n,r,N)$ with $\mathrm{per}(\omega)=0$ which do not have 0-columns.

Thus we have

$$P(A_{nm})=\sum_{j=0}^{m-n}P(A_{nm}\cap C_{nm}^{(j)})+\sum_{j=m-n+1}^{m-1}P(A_{nm}\cap C_{nm}^{(j)})$$

$$=\frac{\sum_{j=0}^{m-n}\binom{m}{j}\delta_{n,m-j}}{|R(n,m,N)|}+\frac{\sum_{j=m-n+1}^{m-1}\binom{m}{j}\gamma_{n,m-j}}{|R(n,m,N)|}.$$

Therefore we obtain

$$P(A_{nm}^{c})=1-P(A_{nm})$$

$$=\frac{\sum_{j=0}^{m-1}\binom{m}{j}\gamma_{n,m-j}}{|R(n,m,N)|}-\frac{\sum_{j=m-n+1}^{m-1}\binom{m}{j}\gamma_{n,m-j}}{|R(n,m,N)|}-\frac{\sum_{j=0}^{m-n}\binom{m}{j}\delta_{n,m-j}}{|R(n,m,N)|}$$

$$=\frac{\sum_{j=0}^{m-n}\binom{m}{j}(\gamma_{n,m-j}-\delta_{n,m-j})}{|R(n,m,N)|}$$

$$\geq\frac{\gamma_{n,m}-\delta_{n,m}}{|R(n,m,N)|}\to 1.$$

Here we used lemma 3 and 4.

THEOREM 6. *Let P be the uniform distribution on $C(n,m,N)$. If $N=\frac{nm}{2}$ and $m=o\left(\exp\frac{n}{2}\right)$, then*

$$P\{\omega\in C(n,m,N)\mid \mathrm{per}(\omega)>0\}\to 1.$$

Proof. Clearly if the permanent of a matrix is positive then, by König-Frobenius theorem, the matrix does not have 0-rows. Hence we have

$$\begin{aligned}&P\{\omega\in C(n,m,N)\mid \mathrm{per}(\omega)>0\}\\&=\frac{|\{\omega\in R(n,m,N)\mid \mathrm{per}(\omega)>0\}|}{|C(n,m,N)|}\\&=\frac{|\{\omega\in R(n,m,N)\mid \mathrm{per}(\omega)>0\}|}{|R(n,m,N)|}\cdot\frac{|R(n,m,N)|}{|C(n,m,N)|}\\&\to 1.\end{aligned}$$

Here we used theorem 5 and lemma 1.

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Kook Min University
Seoul 136-702, Korea,
Korea Advanced Institute of Science and Technology
Seoul 130-010, Korea
and
Electronics and Telecommunications Research Institute
Taejeon 302-338, Korea