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ON STARLIKENESS AND CONVEXITY OF CERTAIN CLASS OF UNIVALENT FUNCTIONS

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In this paper, we show that the radius of convexity for the classes $L(\alpha,\beta,\gamma)$ and $L^*(\alpha,\beta,\gamma)$, and the starlikeness for the classes $L^*(\alpha,\beta,\gamma)$.

1. Introduction

Let S denote the class of the functions the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the unit disk $U=\{z : |z|<1\}$. Let T denote the subclass of S, consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n. \tag{1.2}$$

Let $D(\beta)$ denote the subclass of S consisting of all functions f satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<\beta \qquad (z\in U) \tag{1.3}$$

for some $\beta(0<\beta\leqq 1)$, $G(\beta)$ denote the subclass of S consisting of all functions f satisfying the condition

$$|f'(z)-1|<\beta \qquad (z\in U) \tag{1.4}$$

for some $\beta(0<\beta\leqq 1)$ and let $P(\alpha)$ denote the subclass of S consisting of all functions f satisfying the condition

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$$\text{(1.5)} \qquad \operatorname{Re} f'(z) > \alpha \qquad (z \in U)$$

for some $\alpha(0 \leqq \alpha < 1)$.

The classes $D(\beta)$, $G(\beta)$ and $P(\alpha)$ were studied by Owa[4], Padmanabhan[5] and Singh[6].

Let S^* denote the subclass of S consisting of all functions f satisfying the condition

$$\text{(1.6)} \qquad \operatorname{Re}(zf'(z)/f(z)) > 0 \qquad (z \in U)$$

and let K denote the subclass of S of all functions f satisfying the condition

$$\text{(1.7)} \qquad \operatorname{Re}(1 + zf''(z)/f'(z)) > 0 \qquad (z \in U)$$

Functions in the class S^* are known as starlike functions and functions in the class K are known as convex.

Let $L(\alpha, \beta, \gamma)$ denote the subclass of S consisting of all functions f satisfying the condition

$$\text{(1.8)} \qquad \left| \frac{f'(z) - 1}{\alpha f'(z) + (1 - \gamma)} \right| < \beta \qquad (z \in U)$$

for some $\alpha(0 \leqq \alpha \leqq 1)$, $\beta(0 < \beta \leqq 1)$ and $\gamma(0 \leqq \gamma < 1)$.

Let $L^*(\alpha, \beta, \gamma) = L(\alpha, \beta, \gamma) \cap T$. $D^*(\beta) = D(\beta) \cap T$, $G^*(\beta) = G(\beta) \cap T$, $P^*(\alpha) = P(\alpha) \cap T$ and $T^* = S^* \cap T$.

The class $L(1, \beta, 0)$ when $\alpha = 1$, $\gamma = 0$ was studied by Padmanabhan [5], the class $L(0, \beta, 0)$ when $\alpha = 0$, $\gamma = 0$ was studied by MacGregor[2] and Singh[10], $L^*(0, 1, 0)$ when $\alpha = 0$, $\beta = 1$, $\gamma = 0$ was studied by Silverman[9].

In this paper, we show that the radius of convexity for $L(\alpha, \beta, \gamma)$ and $L^*(\alpha, \beta, \gamma)$ and the starlikeness of the class $L^*(\alpha, \beta, \gamma)$.

2. Convexity

We begin with the statement of the following result due to Kim and Lee[1].

LEMMA 2.1. *A function $f(z)$ defined by (1.1) is in the class $L(\alpha, \beta, \gamma)$ if and only if*

$$\text{(2.1)} \qquad f'(z) = \frac{1 + (1 - \gamma) z\Phi(z)}{1 - \alpha z\Phi(z)} \qquad (z \in U)$$

where $\Phi(z)$ is analytic, $|\Phi(z)| \leqq \beta$ for some $\alpha(0 \leqq \alpha \leqq 1)$, $\beta(0 < \beta \leqq 1)$ and $\gamma(0 \leqq \gamma < 1)$.

From Lemma 2.1, we have

THEOREM 2.2. *If a function $f(z)$ is in the classs $L(\alpha, \beta, \gamma)$, then $f(z)$ is convex in $|z| < r$, where*

$$r = \frac{B - \sqrt{B^2 - 4\alpha\beta}}{2\alpha\beta}, \quad B = 2 + \alpha - \gamma + \alpha\beta \qquad (\alpha \neq 0) \tag{2.2}$$

$$r = 1/(2-\gamma) \qquad (\alpha = 0) \tag{2.3}$$

Proof. We note that $f(z)$ is in the class $L(\alpha, \beta, \gamma)$ if and only if

$$f'(z) = \frac{1 + (1-\gamma) z\Phi(z)}{1 - \alpha z\Phi(z)} \qquad (z \in U), \tag{2.4}$$

where $\Phi(z)$ is analytic and $|\phi(z)| \leqq \beta$. Now it is well known that

$$|h'(z)| \leqq \frac{1 - |h(z)|^2}{1 - |z|^2} \tag{2.5}$$

for any analytic function $h(z)$, $|h(z)| < 1$, in the unit disk U (cf. [3, pp. 168]). Thus

$$|\Phi'(z)| \leqq \frac{\beta^2 - |\Phi(z)|^2}{\beta(1 - |z|^2)}. \tag{2.6}$$

Accordingly, we have that

$$\begin{aligned} \left| \frac{zf''(z)}{f'(z)} \right| &= \left| \frac{(\alpha + 1 - \gamma) z(\Phi(z) + z\Phi'(z))}{(1 + (1-\gamma) z\Phi(z))(1 - \alpha z\Phi(z))} \right| \\ &\leqq \frac{(\alpha + 1 - \gamma)|z|(\beta|z| + |\Phi(z)|)(\beta - |z\Phi(z)|)}{\beta(1 - |z|^2)(1 - (1-\gamma)|z\Phi(z)|)(1 - \alpha|z\Phi(z)|)} \\ &\leqq \frac{(\alpha + 1 - \gamma)|z|}{(1 - |z|)(1 - \alpha\beta|z|)} \end{aligned} \tag{2.7}$$

The function $f(z)$ will be convex if

$$\alpha\beta|z|^2 - (2 + \alpha - \gamma + \alpha\beta)|z| + 1 > 0. \tag{2.8}$$

Case 1. $\alpha \neq 0$. (2.8) is true for

$$|z| < \frac{B - \sqrt{B^2 - 4\alpha\beta}}{2\alpha\beta}, \quad B = 2 + \alpha - \gamma + \alpha\beta. \qquad (\alpha \neq 0) \tag{2.9}$$

Case 2. $\alpha=0$. (2.8) is true for

(2.10) $$|z|<1/(2-\gamma)$$

Clearly, $G(\beta)=L(0,\beta,0)$ and $D(\beta)=L(1,\beta,0)$. As a special case of Theorem 2.2, we get the result in [2]. Putting $\alpha=0$, $\gamma=0$ in Theorem 2.2, we have

COROLLARY 2.3. *If a function $f(z)$ defined by* (1.1) *is in the class $G(\beta)$ then $f(z)$ is convex in $|z|<1/2$.*

Putting $\alpha=1$, $\beta=0$ in Theorem 2.2, we have

COROLLARY 2.4. *If a function $f(z)$ defined by* (1.1) *is in the class $D(\beta)$ then $f(z)$ is convex in $|z|<1/(3+\beta-\sqrt{9+2\beta+\beta^2})/2\beta$.*

LEMMA 2.5. *A function $f(z)$ defined by* (1.2) *is in the class $L^*(\alpha,\beta,\gamma)$ if and only if*

(2.11) $$\sum_{n=2}^{\infty}(1+\alpha\beta)n|a_n|\leqq\beta(\alpha+1-\gamma).$$

The above result due to Kim and Lee [1]. From Lemma 2.5, we have

THEOREM 2.6. *If a functon $f(z)$ defined by* (1.2) *is in the class $L^*(\alpha,\beta,\gamma)$, then $f(z)$ is convex in $|z|<r$, where*

(2.12) $$r=\inf_{n\geqq 2}\left\{\frac{1+\alpha\beta}{n\beta(\alpha+1-\gamma)}\right\}^{1/(n-1)}.$$

Proof. We note that

(2.13) $$\left|z\frac{f''(z)}{f'(z)}\right|\leqq\frac{\sum_{n=2}^{\infty}n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}}.$$

The function $f(z)$ will be convex if

(2.14) $$\sum_{n=2}^{\infty}n^2|a_n||z|^{n-1}<1.$$

Accordingly to Lemma 2.5, $\sum_{n=2}^{\infty}(1+\alpha\beta)n|a_n|/\beta(\alpha+1-\gamma)<1$.

Hence (2.14) will be true if

$$(2.15) \qquad n|z|^{n-1} < \frac{1+\alpha\beta}{\beta(\alpha+1-\gamma)} \ (n=2, 3, \cdots).$$

Solving (2.15) for $|z|$, we obtain

$$(2.16) \qquad |z| < \left\{\frac{1+\alpha\beta}{n\beta(\alpha+1-\gamma)}\right\}^{1/(n-1)} (n=2, 3, \cdots).$$

Setting $r=\inf_{n\geq 2}\left\{\frac{1+\alpha\beta}{n\beta(\alpha+1-\gamma)}\right\}^{1/(n-1)}$, the result follows.

Putting $\alpha=0$, $\gamma=0$ or $\alpha=1$, $\gamma=0$, we have

COROLLARY 2.7. *If a function $f(z)$ defined by* (1.2) *is in the class $G^*(\beta)$ ($D^*(\beta)$ respectively), then $f(z)$ is convex in $|z|<r$, where*

$$(2.17) \quad r=\inf_{n\geq 2}\left\{\frac{1}{n\beta}\right\}^{1/(n-1)}\left(r=\inf_{n\geq 2}\left\{\frac{1+\beta}{2n\beta}\right\}^{1/(n-1)}, \text{ respectively}\right)$$

for $(0<\beta\leq 1)$.

In [1], Kim and Lee proved that $T=L^*(0,1,0)$ and $P^*(\alpha)=L^*\left(\alpha, \frac{1-\alpha}{1+\alpha^2}, 0\right)$ for $\alpha(0\leq\alpha<1)$. From these facts, we have the following corollaries

COROLLARY 2.8. *If a function $f(z)$ defined by* (1.2) *is in the class T, then $f(z)$ is convex in $|z|<r=1/2$.*

The above result is the case $\alpha=0$ in Theorem 8 in [9].

COROLLARY 2.9. *If a function $f(z)$ defined by* (1.2) *is in the class $P^*(\alpha)$ $(0\leq\alpha<1)$, then $f(z)$ is convex in $|z|<r$, where*

$$r=\inf_{n\geq 2}(1/n(1-\alpha))^{1/(n-1)}.$$

3. Starlikeness

We begin with the statement of fhe following result due to Silverman [9] also due to Singh[10].

LEMMA 3.1. *A function $f(z)$ defined by* (1.2) *is in the class $T^*(\delta)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\delta)|a_n| \leqq 1-\delta. \quad (0 \leqq \delta < 1)$$

From Lemma 2.5 and Lemma 3.1, we have

THEOREM 3.2. *If a function $f(z)$ defined by* (1.2) *is in the class $L^*(\alpha, \beta, \gamma)$, then $f(z)$ is in $T^*(\delta)$, where $\delta=(1+\beta\gamma-\beta)/(1+\alpha\beta)$.*

Proof. From Lemma 2.5, we note that

$$\sum_{n=2}^{\infty} n|a_n| \leqq \frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}$$

and from Lemma 3.1, we get

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\delta.$$

Hence $f(z)$ is in the class $T^*(\delta)$ for $\delta \leqq (1+\beta\gamma-\beta)/(1+\alpha\beta)$.

COROLLARY 3.3. *If a function $f(z)$ defined by* (1.2) *is in the class $P^*(\alpha)$, then $f(z)$ in $T^*(\alpha)$.*

Proof. Assume that $f(z) \in P^*(\alpha) = L^*\left(\alpha, \dfrac{1-\alpha}{1+\alpha^2}, 0\right)$. By Theorem 3.2, we have

$$f(z) \in T^*\left(\frac{1+\beta\gamma-\beta}{1+\alpha\beta}\right) = T^*\left(\frac{1-\dfrac{1-\alpha}{1+\alpha^2}}{1+\alpha\dfrac{1-\alpha}{1+\alpha^2}}\right) = T^*(\alpha).$$

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