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# A NOTE ON THE SUBNORMAL OPERATORS

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## 1. Introduction

In this note, the symbol $H$ will denote a Hilbert space with the inner product $(x, y)$ for $x, y \in H$, an operator $T$ on $H$ is bounded, and we use following notations and terminologies.

The *spectrum*, the *point spectrum*, the *residual spectrum* and *approximate point spectrum* of $T$ are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_a(T)$, respectively.

The *spectral radius* $r(T)$ of $T$ is

$$\lim_{n\to\infty}\|T^n\|^{1/n}=\sup\{|\lambda| : \lambda\in\sigma(T)\}.$$

Let $X$ be a closed set in the complex plane $\boldsymbol{C}$ and let $R(X)$ be the algebra of all rational functions with no poles in $X$. The set $X$ is called a *spectral set* for $T$ if

$$\sigma(T)\subset X$$

and

$$\|f(T)\|\leqq\sup\{|f(\lambda)| : \lambda\in X\}$$

for all $f\in R(X)$.

An operator $T$ is said to be satisfied the *growth condition* with respect to a closed set $X$ which contains $\sigma(T)$ if

$$\|(T-\alpha I)^{-1}\|\leqq\frac{1}{d(\alpha, X)}$$

for $\alpha\notin X$, where $d(\alpha, X)$ is the distance from $\alpha$ to $X$.

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If $X$ is a spectral set for $T$, then $T$ satisfies the growth condition with respect to $X$. In fact, if $X$ is a spectral set for $T$,

$$\|(T-\alpha I)^{-1}\| \leqq \sup\{|(\alpha-\lambda)^{-1}| : \lambda\in X\} = \frac{1}{d(\alpha, X)}$$

for $\alpha\notin X$, by the definition of the spectral sets.

An operator $S$ on a Hilbert space $H$ is *subnormal* if there is a Hilbert space $K$ containing $H$ and a normal operator $N$ on $K$ such that $N$ leaves $H$ invariant and $S=N|_H$. In other words, an operator is subnormal if it has a normal extension.

PROPOSITION 1.1([1]). *If $S$ is a subnormal operator and $N$ is a normal extension of $S$, then*

$$\sigma(N)\subset\sigma(S) \text{ and } \sigma_a(S)\subset\sigma(N).$$

PROPOSITION 1.2([5]). *If $T$ is a normal operator, then $T^*$ is normal, $T+\lambda I$ is also normal for every $\lambda\in \boldsymbol{C}$, and $r(T)=\|T\|$.*

PROPOSITION 1.3([5]). *If $T$ is a normal operator on $H$ and injective, then $T^*$ is injective and $T^{-1}$ is normal.*

## 2. Main results

In this section, we shall investigate some properties of subnormal operators.

LEMMA 2.1. *If $S$ is a subnormal operator, then $\sigma(S)$ is a spectral set for $S$.*

*Proof.* Let $N$ be a normal extension of $S$ and let $f\in R(\sigma(S))$. Then by the spectral mapping theorem and the proposition 1.1,

$$\begin{aligned}\|f(S)\| &\leqq \|f(N)\| \\ &= \sup\{|f(\lambda)| : \lambda\in\sigma(N)\} \\ &\leqq \sup\{|f(\lambda)| : \lambda\in\sigma(S)\}.\end{aligned}$$

Therefore, $\sigma(S)$ is a spectral set for $T$.

LEMMA 2.2. *If $S$ is a subnormal operator, then $S$ satisfies growth condition with respect to $\sigma(S)$.*

*Proof.* Let $N$ be a normal extension of $S$ and let $\lambda \notin \sigma(S)$. Then by propositions 1.2 and 1.3, $(N-\lambda I)^{-1}$ is also normal operator.

Therefore, by the proposition 1.2,

$$\begin{aligned}\|(S-\lambda I)^{-1}\| &\leqq \|(N-\lambda I)^{-1}\| \\ &= r((N-\lambda I)^{-1}) \\ &= \sup\{|\mu| : \mu \in \sigma((N-\lambda I)^{-1})\} \\ &= \sup\{|(\alpha-\lambda)^{-1}| : \alpha \in \sigma(N)\} \\ &\leqq \sup\{|(\alpha-\lambda)^{-1}| : \alpha \in \sigma(S)\} \\ &= \frac{1}{d(\lambda, \sigma(S))}.\end{aligned}$$

Let $T$ be an operator on a Hilbert space $H$ and let $B$ be an operator on a larger Hilbert space $K$ which contains $H$ as a subspace. Then $B$ is called a *dilation* of $T$ and $K$ *is* called a *dilation space* if

$$Tx = PBx$$

for all $x \in H$, where $p$ is the orthogonal projection on $H$.

If $B$ is an unitary operator, then $B$ is called an *unitary dilation* of $T$.

In addition, if $B$ satisfies

$$T^n \dot{x} = PB^n x \quad (n=1, 2, \cdots)$$

for all $x \in$ H, $B$ is called a *strong dilation* of $T$.

A subspace $M$ of $H$ is called an *invariant subspace* for $T$ if $TM \subset M$. A subspace $M$ is a *reducing subspace* for $T$, or reduces $T$, if it is invariant under $T$ and $T^*$.

If the dilation space $K$ does not contain a proper subspace which contains $H$ and reduces, then $B$ is called the *minimal* (*strong*) *dilation* of $T$.

An operator $T$ is called a *contraction* if $\|T\| \leqq 1$.

LEMMA 2.3([2]). *Let $T$ be a contraction operator on a Hilbert space $H$. Then there exists the minimal strong unitary dilation $U$.*

Let $T$ be an operator on a Hilbert space $H$ and $\zeta \in \boldsymbol{C}$. The sets $E_\zeta[T]$ and $A_\zeta[T]$ is defined by

$$E_\zeta[T] = \{x \in H : Tx = \zeta x\}$$

and

$$A_\zeta[T]=\{\{x_n\} : x_n\in H,\ \|x_n\|=1,\ (T-\zeta I)x_n\to 0 \text{ as } n\to\infty\},$$

respectively.

Of cours, $E_\zeta[T]\neq\{0\}$ if and only if $\zeta\in\sigma_p(T)$.

It is well known that

$$E_\zeta[T]=E_{\bar\zeta}[T^*]$$

for every normal operators ([4]). In general,

$$E_\zeta[T]=E_{\bar\zeta}[T^*]$$

does not hold for non-normal operators.

PROPOSITION 2.4. *If $T$ is a contraction operator and $|\zeta|=1$, then we have*

$$E_\zeta[T]=E_{\bar\zeta}[T^*] \text{ and } A_\zeta[T]=A_{\bar\zeta}[T^*].$$

*Proof.* Let $U$ be an unitary dilation of $T$.

Then

$$\begin{aligned}\|Ux-\zeta x\|^2&=\|Ux\|^2+\|\zeta x\|^2-2\mathrm{Re}\,\bar\zeta(Ux,x)\\&=2\|\zeta x\|^2-2\mathrm{Re}\,\bar\zeta(Ux,x)\\&=2\mathrm{Re}(\zeta x,\zeta x-Tx)\\&\leqq 2\|x\|\|(T-\zeta I)x\|\\&\leqq 2\|x\|\|(U-\zeta I)x\|\end{aligned}$$

for all $x\in H$, Thus we have

$$(T-\zeta I)x_n\to 0 \text{ as } n\to\infty \text{ iff } (U-\zeta I)x_n\to 0 \text{ as } n\to\infty$$

for a sequence $\{x_n\}$ of unit vectors of $H$.

Similarly, we have

$$(T^*-\bar\zeta I)x_n\to 0 \text{ as } n\to\infty \text{ iff } (U^*-\bar\zeta I)x_n\to 0 \text{ as } n\to\infty$$

for a sequence $\{x_n\}$ of unit vectors of $H$.

Since $U$ is unitary, $(U-\zeta I)x_n\to 0$ as $n\to\infty$ is equivalent to $(U^*-\zeta I)x_n\to 0$ as $n\to\infty$.

Therefore,

$$A_\zeta[T]=A_{\bar\zeta}[T^*] \text{ and } E_\zeta[T]=E_{\bar\zeta}[T^*].$$

Let $X$ be a closed set of the complex plane $\boldsymbol{C}$.

A point $\lambda \in X$ is called a *semi-bare point* of $X$ if there exists a circle through $\lambda$ such that no points of $X$ lie inside this circle.

PROPOSITION 2.5. *If $S$ is a subnormal operator, then the set of semi-bare points of $\sigma(S)$ does not intersect the residual spectrum $\sigma_r(S)$ of $S$.*

*In addition, if $\zeta \in \sigma(S)$ is a semi-bare point of $\sigma(S)$, then*

$$E_\zeta[S] = E_{\bar{\zeta}}[S^*] \text{ and } A_\zeta[S] = A_{\bar{\zeta}}[S^*].$$

*Proof.* Let $\zeta \in \sigma(S)$ be a semi-bare point of $\sigma(S)$.

Then, by the definition of seme-bare point, there exists a point $\alpha_0 \notin \sigma(S)$ such that

$$|\zeta - \alpha_0| = d(\alpha_0, \sigma(S))$$

In fact, $\alpha_0$ is the center of the circle $C$ through $\zeta$ such that no points of $\sigma(S)$ lie inside $C$.

Let $\{\alpha_n\}$ be a sequence of complex numbers lying on the line segment $\overline{\alpha_0 \circ \zeta}$ such that

$$\frac{1}{\sqrt{n+1}} < |\alpha_n - \zeta| < \frac{1}{\sqrt{n}}.$$

Then, each $D_n = \{\lambda : |\lambda - \alpha_n| < r_n\}$ with $r_n = |\alpha_n - \zeta| - \frac{1}{n+1}$ is contained in the resolvent set $\rho(S)$ of $S$ and

$$\alpha_n \to \zeta, \quad r_n^{-1} |\alpha_n - \zeta| \to 1 \text{ as } n \to \infty.$$

Now, let $\{x_n\}$ be a sequence of unit vectors such that

$$\|(S^* - \bar{\zeta} I) x_n\| \to 0 \text{ as } n \to \infty.$$

Since $\alpha_k \notin \sigma(S)$ for each $k$, $S_k = r_k (S - \alpha_k I)^{-1}$ exists as a bounded operator for each $k$ and

$$\|(S_k^* - \bar{\zeta}_k I) x_n\| \to 0 \text{ as } n \to \infty,$$

where $\zeta_k = r_k (\zeta - \alpha_k)^{-1}$. By the Lemma (2.2),

$$\|S_k\| = r_k \|(S - \alpha_k I)^{-1}\|$$

$$\leqq \frac{r_k}{d(\alpha_k, \sigma(S))}$$
$$\leqq 1,$$

and so, $S_k$ is a contraction.

By the Lemma (2. 3), there exists an unitary dilation $U_k$ of $S_k$. Then,

$$\begin{aligned}\|(S_k-\zeta_k I)x_n\|^2 &= \|P(U_k-\zeta_k I)x_n\|^2 \\ &\leqq \|(U_k-\zeta_k)x_n\|^2 \\ &= \|(U_k^*-\bar{\zeta}_k I)x_n\|^2 \\ &= 1-|\zeta_k|^2-2\mathrm{Re}\,\zeta_k((S_k^*-\bar{\zeta}_k I)x_n, x_n) \\ &\leqq 1-|\zeta_k|^2+2|\zeta_k|\,\|(S_k^*-\bar{\zeta}_k I)x_n\|.\end{aligned}$$

Since $S_k-\zeta_k I=r_k(S-\alpha_k I)^{-1}(\zeta I-S)(\zeta-\alpha_k)^{-1}$, we have

$$\begin{aligned}\|(S-\zeta I)x_n\| &= \|r_k^{-1}(\zeta-\alpha_k)(S-\alpha_k I)(S_k-\zeta_k I)x_n\| \\ &\leqq r_k^{-1}|\zeta-\alpha_k|(\|S\|+|\alpha_k|)\|(S_k-\zeta_k I)x_n\| \\ &\leqq c r_k^{-1}|\zeta-\alpha_k|(1-|\zeta_k|^2+2|\zeta_k|\,\|(S_k^*-\bar{\zeta}_k I)x_n\|)^{1/2} \\ &= c((r_k^{-1}|\zeta-\alpha_k|)^2-1+2r_k^{-1}|\zeta-\alpha_k|\,\|(S_k^*-\zeta_k I)x_n\|)^{1/2}\end{aligned}$$

for some constant $c>0$.

Let $\varepsilon>0$ be given. Then since $r_n^{-1}|\alpha_n-\zeta|\to 1$ as $n\to\infty$, there exists a positive integer $k$ such that

$$0\leqq(r_k^{-1}|\zeta-\alpha_k|)^2-1<\left(\frac{\varepsilon}{2c}\right)^2.$$

For such a $k$ (fixed), there exists an integer $N_1>0$ such that

$$2r_k^{-1}|\zeta-\alpha_k|\,\|(S_k^*-\bar{\zeta}_k I)x_n\|<\left(\frac{\varepsilon}{2c}\right)^2$$

for all $n\geqq N_1$, because $\|(S_k^*-\zeta_k I)x_n\|\to 0$ as $n\to\infty$ and $\{r_k^{-1}|\zeta-\alpha_k|\}$ is bounded.

By above arguments, we have

$$\|(S-\zeta I)x_n\|<\varepsilon$$

for all $n\geqq N_1$.

Since $\varepsilon$ is arbitrary,

$$\|(S-\zeta I)x_n\|\to 0 \text{ as } n\to\infty.$$

Thus, we have

$$A_{\bar{\zeta}}[S^*] \subset A_{\zeta}[S].$$

Similarly,

$$A_{\zeta}[S] \subset A_{\bar{\zeta}}[S^*].$$

Therefore,

$$A_{\zeta}[S] = A_{\bar{\zeta}}[S^*].$$

If we replace $x_n$ by a vector $x$ in the above proof, we can see that

$$E_{\zeta}[S] = E_{\bar{\zeta}}[S^*].$$

To show first part, suppose that $\zeta \in \sigma_r(S)$ and is a semi-bare point of $\sigma(S)$. Then $\bar{\zeta} \in \sigma_p(T^*)$.

Thus, there exists a unit vector $x$ such that

$$S^* x = \bar{\zeta} x.$$

Since $E_{\bar{\zeta}}[S] = E_{\zeta}[S^*]$, we obtains

$$Sx = \zeta x$$

which is a contradiction and so, $\zeta \notin \sigma_r(S)$.

LEMMA 2.6([3]). *Let $S$ be a subnormal operator on a Hilbert space $H$ and* $\mathrm{Re}\, S = \frac{1}{2}(S+S^*)$ *and* $\mathrm{Im}\, S = \frac{1}{2i}(S-S^*)$.

*Then,*

$$\|S^*S - SS^*\| \leqq \frac{2}{\pi} \|\mathrm{Im}\, S\| \text{ meas } \sigma(\mathrm{Re}\, S),$$
$$\mathrm{Re}\, \sigma(S) = \sigma(\mathrm{Re}\, S) \text{ and } \mathrm{Im}\, \sigma(S) = \sigma(\mathrm{Im}\, S),$$

*where* meas $\sigma(\mathrm{Re}\, S)$ *is the one-dimensional Lebesgue measure of* $\sigma(\mathrm{Re}\, S)$.

LEMMA 2.7. *If $S$ is a subnormal operator, then the operator $\alpha S + \beta S^*$ is subnormal for each pair $\alpha, \beta$ of complex numbers.*

*Proof.* Let $N$ be a normal extension of $S$ and let $T = \alpha N + \beta N^*$. Then

$$T^*T - TT^* = (|\alpha|^2 - |\beta|^2)(N^*N - NN^*)$$

and so, $T=\alpha N+\beta N^*$ is a normal extension of $\alpha S+\beta S^*$.

An operator $T$ is called a *polynomially compact operator* if $p(T)$ is compact for some polynomial $p(\lambda)$.

PROPOSITION 2.8. *If $S$ is a subnormal operator such that $T=\alpha S+\beta S^*$ for complex numbers $\alpha$ and $\beta$ with $|\alpha|\neq|\beta|$ is a polynomial compact operator, then $S$ is a normal operator.*

*Proof.* By hypothesis, there exists a polynomial $p(\lambda)$ such that $p(T)$ is a compact operator.

Since the spectrum $\sigma(p(T))$ is at most countable ([1], [5]), it is follows by the spectral mapping theorem that $\sigma(T)$ is also at most countable.

Thus, we have

$$\text{meas}(\sigma(\text{Re } T))=\text{meas Re } \sigma(T)=0.$$

Hence, by the lemma 2.6,

$$\|T^*T-TT^*\|\leqq\frac{2}{\pi}\|\text{Im } T\|\text{meas } \sigma(\text{Re } T)=0.$$

Since $\|T^*T-TT^*\|=(|\alpha|^2-|\beta|^2)(S^*S-SS^*)$ and $|\alpha|\neq|\beta|$, $S$ is a normal operator.

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