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ω-SINGULAR SPECTRUM

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1. Introduction

Beurling[1] and Roumieu[6], [7] generalized the theory of Schwartz distributions around 1960. In Beurling's ultradistribution theory $\log(1+|\xi|)$ is replaced by more general subadditive function $\omega(\xi)$. Then the test function space $\mathscr{D}_\omega$ is smaller than $\mathscr{D}$, and the class of ultradistributions $\mathscr{D}_\omega'$ larger than $\mathscr{D}'$. On the other hand, in Roumieu's ultradistribution theory the test function space C_0^∞ is replaced by the non quasianalytic Denjoy-Carleman class C^L. (See Hörmander[4], [5]). Also the theory of Beurling's ultradistribution was refined in Björck[2] to generalize most of the theorems in Chap. III, IV, VI of Hörmander [3].

In this paper we shall attempt to give an easier treatment of microlocal analysis for somewhat restricted class of Beurling's ultradistributions. This class that we consider here can be slightly different from the class considered in Hörmander[4], [5].

In the next section, following Björck[2] we recall the standard notations, definitions, and theorems to state Paley-Wiener type theorems for Beurling ultradifferentiable functions and ultradistributions. We refer to Björck[2] and Hörmander[4] for the other notations appearing in the next section.

2. Definitions and Paley-Wriner Theorems

We mainly limit ourselves to the following class of concave functions in the sequel.

DEFINITION 2.1. $\Omega(t)$ is said to be a concave function of convergence

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type if $\Omega(t)$ is an increasing continuous concave function on $[0, \infty)$.

The following conditions should be required for the existence of test functions and the partition of unity.

DEFINITION 2.2. We denote by $\mathscr{N}$ the set of all functions ω on $\boldsymbol{R}^n$ such that

(α) $\omega(\xi)=\Omega(|\xi|)$ where Ω is a concave function of convergence type,

(β) $J(\Omega)=\int_1^\infty \frac{\Omega(t)}{t^2}dt<\infty.$

Following Beurling[1] and Björck[2] we now introduce the definition of ultradifferentiable functions.

Let $\omega\in\mathscr{N}$, $\phi\in L^1(\boldsymbol{R}^n)$ and λ be a real number. Then we write

$$\|\phi\|_\lambda=\int |\hat{\phi}(\xi)|e^{\lambda\omega(\xi)}d\xi.$$

DEFINITION 2.3. $\mathscr{D}_\omega$ is the set of all $\phi\in L^1(\boldsymbol{R}^n)$ such that ϕ has compact support and $\|\phi\|_\lambda<\infty$ for all $\lambda>0$. The element of $\mathscr{D}_\omega$ is called a ultradifferentiable function. Also if $E\subset\boldsymbol{R}^n$, then

$$\mathscr{D}_\omega(E)=\{\phi\in\mathscr{D}_\omega;\ \text{supp}\ \phi\subset E\}.$$

Let K be compact and Ω open. Then as in the case of $\mathscr{D}(K)$ and $\mathscr{D}(\Omega)$, $\mathscr{D}_\omega(K)$ is a Fréchet space under the seminorms $\|\cdot\|_m$ ($m=1, 2\cdots$) and $\mathscr{D}_\omega(\Omega)$ can be defined as the inductive limit of Fréchet spaces $\mathscr{D}_\omega(K)$.

If $\omega\in\mathscr{N}$, then Beurling showed that the condition (β), the existence of partitions of unity and the nontrivialness of $\mathscr{D}_\omega$ are all equivalent. Also he showed for $\omega\in\mathscr{N}$ that $\mathscr{D}_\omega(\Omega)\subset C_0^\infty(\Omega)$ for every open set Ω in $\boldsymbol{R}^n$ if and only if for some real a and positive b we have

$$\omega(\xi)\geq a+b\log(1+|\xi|)\ \ {}^\forall\xi\in\boldsymbol{R}^n.$$

Now we are in a position to define the Beurling class $\mathscr{B}$.

DEFINITION 2.4. We denote by $\mathscr{B}$ the set of all continuous real valued functions ω on $\boldsymbol{R}^n$, satisfying conditions (α), (β) and (γ) :

(α) $\omega(\xi)=\Omega(|\xi|)$ where Ω is a concave function of convergence type,

(β) $J(\Omega)=\int_1^\infty \frac{\Omega(t)}{t^2}dt<\infty,$

(γ) there exist real a and positive b such that

$$\omega(\xi)\geq a+b\log(1+|\xi|),\quad \xi\in\boldsymbol{R}^n.$$

We will now state the generalized Paley-Wiener theorem for ultra-differentiable functions.

THEOREM 2.5. *Let $\omega\in\mathscr{B}$ and let K be a compact convex set in $\boldsymbol{R}^n$ with support function H. If U is an entire function of n complex variables $\zeta=\xi+i\eta=(\zeta_1,\cdots,\zeta_n)$, the following three conditions are equivalent:*

(i) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}$ such that*

$$\int_{\boldsymbol{R}^n}|U(\xi+i\eta)|e^{\lambda\omega(\xi)}d\xi\leq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|},\quad \eta\in\boldsymbol{R}^n.$$

(ii) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}'$ such that*

$$|U(\xi+i\eta)|\leq C_{\lambda,\varepsilon}'e^{H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)},\quad \zeta=\xi+i\eta\in\boldsymbol{C}^n.$$

(iii) $U(\zeta)=\int e^{-i\langle x,\zeta\rangle}\phi(x)dx$ *with some $\phi\in\mathscr{D}_\omega(K)$.*

DEFINITION 2.6. $\mathscr{E}_\omega(\Omega)$ is the set of all complex valued functions ϕ in Ω such that if $\psi\in\mathscr{D}_\omega(\Omega)$, then $\psi\phi\in\mathscr{D}_\omega(\Omega)$.

As in the Schwartz distribution theory we now define the Beurling ultradistributions.

DEFINITION 2.7. Let $\omega\in\mathscr{B}$ and let Ω be an open subset of $\boldsymbol{R}^n$. Then $\mathscr{D}_\omega'(\Omega)$ is the space of all continuous linear functionals on $\mathscr{D}_\omega(\Omega)$.

Finally, we need the following generalizations of support and singular support for Paley-Wiener-Schwartz theorem and microlocal analysis in the Beurling ultradistribution theory.

DEFINITION 2.8. Let $\omega\in\mathscr{B}$. If $u\in\mathscr{D}_\omega'(\Omega)$, the support of u (denoted by supp u) is defined as the smallest closed set K such that $u=0$ in $\Omega\cap K^c$.

DEFINITION 2.9. Let ω_1 and $\omega\in\mathscr{B}$. If $u\in\mathscr{D}_{\omega_1}'(\Omega)$ the ω-singular support of u (denoted by sing_ω supp u) is defined as the smallest closed set K such that $u\in\mathscr{E}_\omega(\Omega\cap K^c)$.

We can now state the Paley-Wiener-Schwartz theorem for ultradistributions with compact support.

THEOREM 2.10. *Let $\omega\in\mathscr{B}$ and let K be a compact convex set in $\boldsymbol{R}^n$ and let H be the support function of K. If U is an entire function of n complex variables $\zeta=\xi+i\eta=(\zeta_1,\cdots,\zeta_n)$, the following three conditions are equivalent:*

(a) *For some real λ and all positive ε, there exists a constant $C_{\lambda,\varepsilon}$ such that*

$$\int_{R^n}|U(\xi+i\eta)|e^{-\lambda\omega(\xi)}d\xi\leq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|}.$$

(b) *For some real λ and all positive ε there exists a constant $C_{\lambda,\varepsilon}'$ such that*

$$|U(\xi+i\eta)|\leq C_{\lambda,\varepsilon}'e^{H(\eta)+\varepsilon|\eta|+\lambda\omega(\xi)},\quad \xi+i\eta\in C^n.$$

(c) *U is the Fourier-Laplace transform of some $u\in\mathscr{E}_\omega$ with* $\operatorname{supp} u\subset K$.

3. Main Results

Making use of Paley-Wiener theorem we will define thet ω-singular spectrum for Beurling ulradistributions. We start from the simple consequence of Paley-Wiener theorem.

LEMMA 3.1. *Let $\omega\in\mathscr{B}$ and $v\in\mathscr{E}_\omega'(\boldsymbol{R}^n)$. Then $v\in\mathscr{D}_\omega(\boldsymbol{R}^n)$ if and only if for every positive number λ there exists a constant C_λ such that*

$$|\hat{v}(\xi)|\leq C_\lambda e^{-\lambda\omega(\xi)},\quad \xi\in\boldsymbol{R}^n.$$

Proof. The proof is clear by the equivalence of (ii) and (iii) in Theorem 2.5 (Also see the proof of Theorem 2.10.).

It is clear from Definition 2 that for $v\in\mathscr{E}_\omega'$ $\operatorname{sing}_\omega \operatorname{supp} v$ can be interpreted as the set of points having no neighborhood where v is in $\mathscr{E}_\omega$. Now we introduce the concept of cone which describes the direction of the high frequencies causing singularities.

DEFINITION 3.2. If $v\in\mathscr{E}_\omega'$, then the cone $\Sigma(v)$ is the set of all $\eta\in\boldsymbol{R}^n\backslash 0$ having no conic neighborhood V such that for every $\lambda>0$ there exists C_λ with

$$|\hat{v}(\xi)|\leq C_\lambda e^{-\lambda\omega(\xi)},\quad \xi\in V.$$

It is clear that $\Sigma(v)$ is a closed cone in $\boldsymbol{R}^n\backslash 0$. Using the following theorem we can combine the information about the location of the singularities and the singularity cone $\Sigma(v)$.

THEOREM 3.2. *If* $\phi\in\mathscr{D}_{\omega}(\boldsymbol{R}^{n})$ *and* $v\in\mathscr{E}_{\omega}'(\boldsymbol{R}^{n})$ *then*

$$\Sigma(\phi v)\subset\Sigma(v).$$

Proof. The Fourier transform of $u=\phi v$ is the convolution

$$\hat{u}(\xi)=\frac{1}{(2\pi)^{n}}\int\hat{\phi}(\eta(\hat{v}(\xi-\eta)d\eta.$$

By Theorem 2.10 there exist some real λ_1 and a constant C_1 such that

$$|\hat{v}(\xi)|\leq C_1e^{\lambda_1\omega(\xi)},\quad \xi\in\boldsymbol{R}^{n}.$$

Let us split the integral into two parts where $|\eta|<\frac{1}{2}|\xi|$ and $|\eta|\geq\frac{1}{2}|\xi|$. Note that $|\xi-\eta|\leq 3|\eta|$ in the second case. Hence

$$\begin{aligned}(2\pi)^{n}|\hat{u}(\xi)|&\leq\int_{|\eta|<1/2|\xi|}|\hat{\phi}(\eta)\hat{v}(\xi-\eta)|d\eta\\&+\int_{|\eta|\geq1/2|\xi|}|\hat{\phi}(\eta)\hat{v}(\xi-\eta)|d\eta\\&=\mathrm{I}+\mathrm{II}.\end{aligned}$$

Then

$$\mathrm{I}\leq\sup_{|\eta-\xi|<1/2|\xi|}|\hat{v}(\eta)|\,\|\hat{\phi}\|_{L^{1}}.$$

It follows from Theorem 2.10 that

$$\begin{aligned}\mathrm{II}&\leq C_1\int_{|\eta|\geq1/2|\xi|}|\hat{\phi}(\eta)|e^{\lambda\omega(\xi-\eta)}d\eta\\&\leq C_1\int_{|\eta|\geq1/2|\xi|}|\hat{\phi}(\eta)|e^{3\lambda\omega(\eta)}d\eta.\end{aligned}$$

Note that the concavity and the increase of ω is essential in the above inequalities. Thus

$$(2\pi)^{n}|\hat{u}(\xi)|\leq\sup_{|\eta-\xi|<1/2|\xi|}|\hat{u}(\eta)|\,\|\hat{\phi}\|_{L^{1}}+C_1\int_{|\eta|\geq1/2|\xi|}|\hat{\phi}(\eta)|e^{3\lambda\omega(\eta)}d\eta.$$

Let Γ be an open cone where

$$|\hat{v}(\xi)|\leq C_\lambda e^{-\lambda\omega(\xi)}$$

i.e., $\Gamma=(\Sigma(v))$ and let $\Gamma_1\subset\Gamma\cup\{0\}$ be a closed cone. Then we can choose $0<c<1$ so that $\eta\in\Gamma$ if $\xi\in\Gamma_1$ and $|\xi-\eta|\leq c|\xi|$. Since $|\xi|-|\eta|\leq|\xi-\eta|\leq c|\xi|$, we have $|\xi|\leq\dfrac{1}{1-c}|\eta|$.

Therefore we have for $\lambda>0$

$$\begin{aligned}(2\pi)^n\sup_{\Gamma_1} e^{\lambda\omega(\xi)}|\hat{u}(\xi)| &\leq \sup_{\eta\in\Gamma}|\hat{v}(\eta)|\,\|\phi\|_{L^1}e^{\lambda\omega(\eta/1-c)}\\ &\quad +C_1\int_{|\eta|>1/2|\xi|}|\hat{\phi}(\eta)|e^{3\lambda\omega(\eta)}e^{\lambda\omega(\eta/1-c)}d\eta\\ &=\mathrm{III}+\mathrm{IV}.\end{aligned}$$

Then

$$\mathrm{III}\leq\sup_{\eta\in\Gamma}|\hat{v}(\eta)|e^{([1/1-c]+1)\lambda\omega(\eta)}\|\phi\|_{L^1},$$

where [ ] denotes the Gauss symbol, and

$$\mathrm{IV}\leq C_1\int|\hat{\phi}(\eta)|e^{3\lambda(\omega(\eta)+\omega(1/3(1-c)\eta))}d\eta.$$

Set

$$k=\frac{1}{3(1-c)}.$$

We consider the following two cases.

i) $k<1$, i.e., $c<\dfrac{2}{3}$

Since ω is increasing

$$\mathrm{IV}\leq C_1\int|\hat{\phi}(\eta)|^{6\lambda\omega(\eta)}d\eta.$$

ii) $k\geq1$, i.e., $c\geq\dfrac{2}{3}$

Similarly,

$$\mathrm{IV}\leq C_1\int|\hat{\phi}(\eta)|e^{6\lambda\omega(([k]+1)\eta)}d\eta.$$

For case i) we have for every $\lambda\geq0$

$$(2\pi)^n \sup_{\Gamma_1} |\hat{u}(\xi)| e^{\lambda\omega(\xi)}$$

$$\leq \sup_{\eta\in\Gamma} |\hat{v}(\eta)| e^{([1/1-c]+1)\lambda\omega(\eta)} \|\phi\|_{L^1} + C_1 \int |\hat{\phi}(\eta)| e^{6\lambda\omega(\eta)} d\eta.$$

It remains to show that the right hand side of the above inequality is finite. Since

$$|\hat{v}(\eta)| \leq C_\mu e^{-\mu\omega(\eta)} \text{ for every } \mu>0,$$

the first term is finite. For the second term let $A=\dfrac{n+1}{b}$. Then because of condition (γ) in Definition 2. 4

$$\int |\hat{\phi}(\eta)| e^{6\lambda\omega(\eta)} d\eta \leq \int |\hat{\phi}(\eta)| e^{(6\lambda+A)\omega(\eta)} e^{-\frac{n+1}{b}\omega(\eta)} d\eta$$

$$\leq \sup_{\eta\in R^n} |\hat{\phi}(\eta)| e^{(6\lambda+A)\omega(\eta)} \int e^{-\frac{n+1}{b}\omega(\eta)} d\eta < \infty.$$

The case ii) can be proved similarly.

Let Ω be an open set in $\boldsymbol{R}^n$ and $u\in\mathscr{D}_\omega'(\Omega)$. We set for $x\in\Omega$.

$$\Sigma_x u = \bigcap_\phi \Sigma(\phi u), \quad \phi\in\mathscr{D}_\omega(\Omega), \quad \phi(x)\neq 0.$$

THEOREM 3. 3. *If* $\phi\in\mathscr{D}_\omega(\Omega)$, $\phi(x)\neq 0$ *and* supp $\phi\to\{x\}$ *then*

$$\Sigma(\phi u)\to\Sigma_x(u).$$

Proof. Let V be an open cone containing $\Sigma_x(u)$. Then the existence of local unit and the compactness of unit sphere allow us to find $\phi_1, \cdots, \phi_j\in\mathscr{D}_\omega(\Omega)$ with

$$\phi_1(x)\cdots\phi_j(x)\neq 0,$$

$$\bigcap_1^j \Sigma(\phi_i u)\subset V.$$

When $\phi\in\mathscr{D}_\omega(\Omega)$ and supp ϕ is so close to x that $\phi_1\cdots\phi_j\neq 0$ there, we can write $\phi=\psi\phi_1\cdots\phi_j$ with $\psi\in\mathscr{D}_\omega(\Omega)$. Also it follows from Theorem 3. 2

$$\Sigma(\phi u)\subset\bigcap_1^j \Sigma(\phi_i u)\subset V.$$

Since $\Sigma(\phi u)\supset\Sigma_x(u)$ when $\phi(x)\neq 0$ by definition the theorem is proved.

Finally we are now in a position to define ω-singular spectrum.

DEFINITION 3.4. If $u\in\mathscr{D}_\omega'(\Omega)$, then the closed subset of $\Omega\times(\boldsymbol{R}^n\backslash 0)$ defined by

$$\text{sing}_\omega \text{ spec } u=\{(x,\xi)\in\Omega\times(\boldsymbol{R}^n\backslash 0)\,;\ \xi\in\Sigma_x(u)\}$$

is called ω-singular spectrum of u. Because of this definition it is clear that its projection in Ω is sing_ω supp u.

REMARK. Most of the theorems on singular spectrum can be directly generalized for ω-singular spectrum. We leave this to the reader.

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