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A CENTRAL LIMIT THEOREM FOR ASSOCIATED RANDOM VECTORS

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1. Introduction

A finite family $\{Y_1, \cdots, Y_m\}$ of random variables is said to be associated if for any real (coordinatewise) nondecreasing functions f and g on R^m

$$\mathrm{Cov}(f(Y_1, \cdots, Y_m), g(Y_1, \cdots, Y_m)) \geq 0,$$

whenever this covariance exists.

Infinite families are associated if every finite subfamily is associated. This definition is due to Esary, Proschan, and Walkup (1967).

There are two almost independent bodies of literature on the subjects of associated random variables. One developed from the work of Esary, Proschan, and Walkup (1967) and Sarkar (1969) and is oriented towards reliability theory and statistics; the other developed from the work of Harris (1960) and of Fortuin, Kastelyn and Ginibre (1971) and is oriented towards percolation theory and statistical mechanics.

It should be noted that in the latter literature, the term 'associated' is usually not used but rather variables are said to satisfy the FKG inequalities.

Associated sequences of random variables have been studied extensively in recent years. Under some restrictions on covariance a wide number of limit theorems have been proved for associated sequences of random variables. Newman (1980) proved the central limit theorem for associated sequences of random variables and he showed a general central limit theorem for associated random variables in 1983.

In section 2 we introduce a generalization of association to R^d-valued random vectors and find a characteristic function inequality for associated

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random vectors using ideas of Newman (1983). With this inequality a central limit theorem for associated random vectors is proved in section 3.

2. A characteristic function inequality

DEFINITION 2.1. A sequence $\{X_i=(X_{i1},\cdots,X_{id}),\ i\geq 1\}$ of R^d-valued random vectors is said to be associated if for all coordinatewise non-decreasing functions f, g on R^{nd}

$$\mathrm{Cov}(f(X_1,\cdots,X_n),\ g(X_1,\cdots,X_n))\geq 0,$$

whenever this covariance is defined.

Let $S_n=\sum_{i=1}^{n}X_i$ denote the partial sum of the sequence $\{X_i,\ i\geq 1\}$. We denote the j th coordinate of the vector X_i by X_{ij}, $1\leq j\leq d$ and in the same way we put $S_{nj}=\sum_{i=1}^{n}X_{ij}$. Furthermore, we define for a sequence $\{X_i, i\geq 1\}$ of stationary associated R^d-valued random vectors with mean zero and finite second moment:

$$\sigma_{ij}(n)=\frac{1}{n}E(S_{ni}S_{nj})$$

$$=E(X_{1i}X_{1j})+\frac{1}{n}\sum_{k=2}^{n}(n-k+1)(EX_{1i}E_{kj}+EX_{ki}X_{1j}),$$

$$\Sigma_n=(\sigma_{ij}(n))_{1\leq i,j\leq d} \tag{2.1}$$

and

$$A_{ij}=E(X_{1i}X_{1j})+\sum_{k=2}^{\infty}(EX_{1i}X_{kj}+EX_{ki}X_{1j}),\quad A=(A_{ij})_{1\leq i,j\leq d}$$

We observe that all the entries in the infinite sum defining A_{ij} are non-negative, hence A_{ij} is always defined and we will assume A_{ij} is finite.

Note that $\{\sigma_{ij}(n)\}$ is increasing and $\sigma_{ij}(n)\leq A_{ij}<\infty$, by (2.1), and association.

A sequence $\{X_i,\ i\geq 1\}$ of random vectors satisfies stationarity (translation invariance) if for all m and for all $j, k_1, k_2, \cdots, k_m\in R^d$, $(X_{k_1},\cdots,X_{k_m})$ has the same distribution as $(X_{j+k_1},\cdots,X_{j+k_m})$.

Newman's Inequality (Newman, 1980, 1981). Suppose $Y_1, \cdots, Y_n$ are associated random variables with finite variance; then for any real $r_1, \cdots, r_n$

$$\left|E[\exp(i\sum_{k=1}^{n} r_k Y_k)] - \prod_{k=1}^{n} E[\exp(ir_k Y_k)]\right| \leq \sum_{1\leq j<k\leq n} |r_j r_k| \operatorname{Cov}(Y_j, Y_k) \tag{2.2}$$

According to Newman (1984, page 134; 1983, page 77) for f and f_1 complex functions on R^m, we write $f \ll f_1$ if f_1-$\operatorname{Re}(\exp(i\alpha f))$ is coordinatewise nondecreasing for all α. Note first that $f_1 = (f_1 - \operatorname{Re}(f)) + (f_1 - \operatorname{Re}(-f))/2$ and hence is automatically nondecreasing and next that $f \ll f_1$ for real f if and only if $f_1 + f$ and $f_1 - f$ are both nondecreasing.

PROPOSITION 2.2. (Newman, 1983, Proposition 1; 1984, Proposition 15). *Suppose that $f \ll f_1$ and $g \ll g_1$ and that $X_1, X_2, \cdots$ are associated; then*

$$\operatorname{Cov}(f_1, g_1) \geq \begin{cases} |\operatorname{Cov}(f, g)|, & \text{if } f \text{ or } g \text{ is real} \quad (2.3a) \\ |\operatorname{Cov}(f, g)|/2, & \text{otherwise}, \quad (2.3b) \end{cases}$$

and

$$\operatorname{Cov}(f_1, g_1) \geq |\operatorname{Cov}(\exp(if), \exp(ig))|/2, \quad \text{if } f \text{ and } g \text{ are real.} \tag{2.4}$$

Using (2.4) of Proposition 2.2, we reduce following lemma which allows us to extend Newman's Inequality (2.2) to associated random vectors at the cost of an additional factor of 2 on the right hand side.

LEMMA 2.3. *Suppose X_1 and X_2 are associated R^d-valued random vectors. Then for any $r_1, r_2 \in R^d$*

$$|\phi(r_1, r_2) - \phi_1(r_1)\phi_2(r_2)| \leq 2\sum_{i=1}^{d}\sum_{j=1}^{d} |r_{1i} r_{2j}| \operatorname{Cov}(X_{1i}, X_{2j}), \tag{2.5}$$

where

$$\phi(r_1, r_2) = E(\exp\{ir_1 X_1\}\exp\{ir_2 X_2\}),$$
$$\phi_j(r_j) = E(\exp\{ir_j X_j\}), \quad j = 1, 2,$$
$$r_j X_j = \sum_{k=1}^{d} r_{jk} X_{jk}.$$

Proof. Let

$$f^+=\sum_{j=1}^d r_{1j}^+X_{1j},\quad f^-=\sum_{j=1}^d r_{1j}^-X_{1j}$$

$$g^+=\sum_{k=1}^d r_{2k}^+X_{2k},\quad g^-=\sum_{k=1}^d r_{2k}^-X_{2k} \tag{2.6}$$

and let

$$f=f^+-f^-,\ f_1=f^++f^-;\ g=g^+-g^-,\ g_1=g^++g^-, \tag{2.7}$$

where for any s we define $s^+=\max(s,0)$ and $s^-=\max(-s,0)$.
Then

$$f_1=\sum_{j=1}^d |r_{1j}|X_{1j},\quad g_1=\sum_{k=1}^d |r_{2k}|X_{2k} \tag{2.8}$$

Since all four functions f^+, f^-, g^+ and g^- are coordinatewise nondecreasing by the notion before Proposition 2.2 we have

$$f\ll f_1 \text{ for real } f,\quad g\ll g_1 \text{ for real } g. \tag{2.9}$$

(2.6) and (2.7) yield

$$\begin{aligned}
&|\phi(r_1,r_2)-\phi_1(r_1)\phi_2(r_2)|\\
&=|E(\exp\{ir_1X_1\}\ \exp\{ir_2X_2\})-E\exp\{ir_1X_1\}E\exp\{ir_2X_2\}|\\
&=|\mathrm{Cov}(\exp\{ir_1X_1\},\ \exp\{ir_2X_2\})|\\
&=|\mathrm{Cov}(\exp\{i\sum_{j=1}^d r_{1j}X_{1j}\},\ \exp\{i\sum_{k=1}^d r_{2k}X_{2k}\})|\\
&=|\mathrm{Cov}(\exp\{i(\sum_{j=1}^d r_{1j}^+X_{1j}-\sum_{j=1}^d r_{1j}^-X_{1j})\},\ \exp\{i(\sum_{k=1}^d r_{2k}^+X_{2k}-\sum_{k=1}^d r_{2k}^-X_{2k})\})|\\
&=|\mathrm{Cov}(\exp\{i(f^+-f^-)\},\ \exp\{i(g^+-g^-)\})|\\
&=|\mathrm{Cov}(\exp(if),\ \exp(ig))|=[Q].
\end{aligned}$$

(2.4) of Proposition 2.2 and (2.9) provide

$$\begin{aligned}
[Q]&\leq 2\mathrm{Cov}(f_1,g_1)\\
&=2\mathrm{Cov}(\sum_{j=1}^d |r_{1j}|X_{1j},\ \sum_{k=1}^d |r_{2k}|X_{2k})
\end{aligned}$$

$$=2\sum_{j=1}^{d}\sum_{k=1}^{d}|r_{1j}r_{2k}|\operatorname{Cov}(X_{1j}, X_{2k}). \tag{2.10}$$

which completes the proof.

THEOREM 2.4. Let $\{X_i=(X_{i1}, \cdots, X_{id}),\ i\geq 1\}$ *be a sequence of stationary associated* R^d*-valued random vectors. Let* ϕ_i *be the characteristic function of* X_i *and let* ϕ^N *be the joint characteristic function of* $X_1, \cdots, X_N$. *Then for any vectors* $r_1, \cdots, r_N \in R^d$ *we have*

$$|\phi^N(r_1, \cdots, r_N)-\prod_{i=1}^{N}\phi_i(r_i)|$$
$$\leq 2\sum_{1\leq k<m\leq N}\ \sum_{1\leq i, j\leq d}|r_{ki}\ r_{mj}|\operatorname{Cov}(X_{ki}, X_{mj}) \tag{2.11}$$

Proof. (2.11) follows from (2.5) by induction on N. The first step of the induction argument to prove (2.11) is true for $N=2$ by Lemma 2.3. Assume that (2.11) holds for $N-1$. It remains to show that this argument is true for N. Using the triangle inequality and Lemma 2.3 we obtain

$$|\phi^N(r_1, \cdots, r_N)-\prod_{i=1}^{N}\phi_i(r_i)|$$
$$\leq|\phi^N(r_1, \cdots, r_N)-\phi^{N-1}(r_1, \cdots, r_{N-1})\phi_N(r_N)|$$
$$+|\phi^{N-1}(r_1, \cdots, r_{N-1})\phi_N(r_N)-\prod_{i=1}^{N}\phi_i(r_i)|$$
$$=\operatorname{Cov}(\exp\{i(\sum_{k=1}^{N-1}\sum_{i=1}^{d}r_{ki}{}^{+}X_{ki}-\sum_{k=1}^{N-1}\sum_{i=1}^{d}r_{ki}{}^{-}X_{ki})\},$$
$$\exp\{i(\sum_{j=1}^{d}r_{Nj}{}^{+}X_{Nj}-\sum_{j=1}^{d}r_{Nj}{}^{-}X_{Nj})\})$$
$$+|\phi^{N-1}(r_1, \cdots, r_{N-1})-\prod_{i=1}^{N-1}\phi_i(r_i)|\,|\phi_N(r_N)|$$
$$\leq 2\operatorname{Cov}(\sum_{k=1}^{N-1}\sum_{i=1}^{d}(r_{ki}{}^{+}+r_{ki}{}^{-})X_{ki},\ \sum_{j=1}^{d}(r_{Nj}{}^{+}+r_{Nj}{}^{-})X_{Nj})$$
$$+|\phi^{N-1}(r_1, \cdots, r_{N-1})-\prod_{i=1}^{N-1}\phi_i(r_i)|$$

$$=2\sum_{k=1}^{N-1}\sum_{1\le i,j\le d}|r_{ki}r_{Nj}|\mathrm{Cov}(X_{ki}, X_{Nj})$$

$$+|\phi^{N-1}(r_1, \cdots, r_{N-1})-\prod_{i=1}^{N-1}\phi_i(r_i)|$$

$$=2\sum_{1\le k<m\le N}\sum_{i,j=1}^{d}|r_{ki}r_{mj}|\mathrm{Cov}(X_{ki}, X_{mj}).$$

3. A central limit theorem

The next theorem gives us a central limit theorem for associated random vectors, which is an extension of a central limit theorem for associated random variables (Newman, 1980).

THEOREM 3.1. *Suppose* $\{X_i, i\ge 1\}$ *is a sequence of stationary mean zero, finite second moment,* R^d*-valued random vectors which are associated and such that*

(i) $\sigma_{ij}(n)\to A_{ij}$ *as* $n\to\infty$ *for all* $i, j\,(1\le i,\ j\le d)$, $A_{ij}<\infty$

(ii) $E\left|\frac{S_n}{\sqrt{n}}\right|^2<\infty$;

then $(1/\sqrt{n})S_n$ *converges to normal law with nondegenerated covariance matrix* A.

Proof. It is sufficient to show that for all vector $\lambda=(\lambda_1, \cdots, \lambda_d)\in R^d$,

$$E\exp\{i(\lambda, S_n)\}\to\exp((-\lambda A\lambda^t)/2) \quad \text{as } n\to\infty, \tag{3.1}$$

where $(\lambda, S_n)=\lambda_1 S_{n1}+\lambda_2 S_{n2}+\cdots+\lambda_d S_{nd}$ and

λ^t is the transposed matrix of λ.

If we define $m(n)=[n/k]$, then we have that $m\to\infty$ as $n\to\infty$. For any $\lambda=(\lambda_1, \cdots, \lambda_d)\in R^d$, define

$$Y_i=\sum_{j=1}^{d}X_{ij}\lambda_j. \tag{3.2}$$

Thus (3.1) becomes

$$\left|E\exp\{i(\lambda, S_n/\sqrt{n})\}-\exp\left(-\frac{1}{2}\lambda A\lambda^t\right)\right|$$

$$=\left|E\exp\{i(\sum_{j=1}^{d}S_{nj}\lambda_j/\sqrt{n})\}-\exp\left(-\frac{1}{2}\lambda A\lambda^t\right)\right|$$

$$= \left| E\exp\{i(\sum_{j=1}^{d}\sum_{\nu=1}^{n} X_{\nu j}\lambda_j/\sqrt{n})\} - \exp\left(-\frac{1}{2}\lambda A\lambda^t\right)\right|$$

$$= \left| E\exp\{i\sum_{\nu=1}^{n} Y_\nu/\sqrt{n}\} - \exp\left(-\frac{1}{2}\lambda A\lambda^t\right)\right|$$

as $n\to\infty$. (3.3)

Using triangle inequality (3.3) is bounded by (I) + (II) + (III);

$$(\mathrm{I}) = |E\exp\{i\sum_{\nu=1}^{n} Y_\nu/\sqrt{n}\} - E\exp\{i\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk}\}|,$$

$$(\mathrm{II}) = |E\exp\{i\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk}\} - \prod_1^m E\exp\{im^{-1/2}(k^{-1/2}\sum_{\nu=1}^{k} Y_\nu)\}|,$$

$$(\mathrm{III}) = \left|\prod_1^m E\exp\{im^{-1/2}(k^{-1/2}\sum_{\nu=1}^{k} Y_\nu)\} - \exp\left(-\frac{1}{2}\lambda A\lambda^t\right)\right|.$$

Now it remains to show that all (I), (II), and (III) converge to zero as n goes infinity. In order to prove this we consider it as following three steps:

First step; by the inequality $|\exp(ix)-1|\le|x|$ and Cauchy-Schwarz's inequality,

$$(\mathrm{I}) = |E\exp\{i\sum_{\nu=1}^{n} Y_\nu/\sqrt{n}\} - E\exp\{i\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk}\}|$$

$$= |E[\exp\{i\sum_{\nu=1}^{n} Y_\nu/\sqrt{n}\}(1-\exp\{i(\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk} - \sum_{\nu=1}^{n} Y_\nu/\sqrt{n})\})]|$$

$$\le E|1-\exp\{i(\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk} - \sum_{\nu=1}^{n} Y_\nu/\sqrt{n})\}|$$

$$\le E|\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk} - \sum_{\nu=1}^{n} Y_\nu/\sqrt{n}|$$

$$\le E|\sum_{\nu=1}^{mk}(Y_\nu/\sqrt{mk} - Y_\nu/\sqrt{n})| + E|\sum_{\nu=mk+1}^{n} Y_\nu/\sqrt{n}|, \quad \text{since}$$

$$m=\left[\frac{n}{k}\right] \text{ i.e. } mk\le n$$

$$\le (E|\sum_{\nu=1}^{mk} Y_\nu/\sqrt{mk}|^2)^{1/2}\left(1-\sqrt{\frac{mk}{n}}\right)$$

$$+(E|\sum_{\nu=mk+1}^{n}Y_\nu/\sqrt{n-mk}|^2)^{1/2}\sqrt{1-\frac{mk}{n}}\to 0$$

as $n\to\infty$. (3.4)

Next step; let

$$Z_\gamma=k^{-1/2}\sum_{\mu=1}^{k}X_{k(\gamma-1)+\mu}. \tag{3.5}$$

Then (3.2) and (3.5) give us

$$\begin{aligned}\sum_{\nu=1}^{mk}Y_\nu/\sqrt{mk}&=\sum_{\nu=1}^{m}m^{-1/2}\sum_{\mu=1}^{k}k^{-1/2}Y_{k(\nu-1)+\mu}\\ &=\sum_{\nu=1}^{m}m^{-1/2}\sum_{\mu=1}^{k}k^{-1/2}(\sum_{i=1}^{d}X_{k(\nu-1)+\mu i}\lambda_i)\\ &=\sum_{\nu=1}^{m}m^{-1/2}\sum_{i=1}^{d}Z_{\nu i}\lambda_i \end{aligned}\tag{3.6}$$

and

$$\begin{aligned}k^{-1/2}\sum_{\mu=1}^{k}Y_\mu&=k^{-1/2}\sum_{\mu=1}^{k}\sum_{j=1}^{d}X_{\mu j}\lambda_j\\ &=\sum_{j=1}^{d}Z_{1j}\lambda_j.\end{aligned}\tag{3.7}$$

On the other hand from the stationarity of $\{X_i,\ i\geq 1\}$

$$\begin{aligned}E\exp\{i(\lambda/\sqrt{m})Z_1\}&=E\exp\{i(\lambda/\sqrt{m})Z_2\}=\cdots\\ &=E\exp\{i(\lambda/\sqrt{m})Z_m\}.\end{aligned}\tag{3.8}$$

We apply Theorem 2.4 with $r_1=r_2=r_3=\cdots=r_m=\lambda/\sqrt{m}\in R^d$, and combine (3.6), (3.7), and (3.8) to obtain the following bound on (II);

$$\begin{aligned}(\text{II})&=|E\exp\{i\sum_{\nu=1}^{m}m^{-1/2}\sum_{i=1}^{d}Z_{\nu i}\lambda_i-\prod_{1}^{m}E\exp\{im^{-1/2}\sum_{i=1}^{d}Z_{1i}\lambda_i\}|\\ &=|E\exp\{i\sum_{\nu=1}^{m}(\lambda/\sqrt{m})Z_\nu\}-\prod_{1}^{m}E\exp\{i(\lambda/\sqrt{m})Z_1\}|\\ &=|E\exp\{i\sum_{\nu=1}^{m}(\lambda/\sqrt{m})Z_\nu\}-\prod_{\nu=1}^{m}E\exp\{i(\lambda/\sqrt{m})Z_\nu\}|\end{aligned}$$

$$\leq 2 \sum_{1\leq \gamma<\nu\leq m} \sum_{i,j=1}^{d} |(\lambda_i\lambda_j/m)| \operatorname{Cov}(Z_{\gamma i}, Z_{\nu j})$$

$$\leq 2\|\lambda\|^2 \sum_{1\leq \gamma<\nu\leq m} \sum_{i,j=1}^{d} m^{-1}\operatorname{Cov}(Z_{\gamma i}, Z_{\nu j}),$$

where $\|\lambda\|^2=\lambda_1^2+\cdots+\lambda_d^2.$ (3.9)

Generally

$$\sum_{1\leq i<j\leq m} \operatorname{Cov}(X_i, X_j)=\operatorname{Cov}(\sum_{i=1}^{m} X_i, \sum_{i=1}^{m} X_i)-\sum_{i=1}^{m}\operatorname{Cov}(X_i, X_i) \quad (3.10)$$

holds.

From the stationarity of $\{X_i, i\geq 1\}$ we have

$$\operatorname{Cov}(\sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i}, \sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i})$$

$$=\operatorname{Cov}(\sum_{\mu=1}^{k} X_{\mu i}, \sum_{\mu=1}^{k} X_{\mu i}). \quad (3.11)$$

Applying (3.5), (3.10), (3.11), and (2.1) to (3.9) we obtain

$$2\|\lambda\|^2 \sum_{i,j=1}^{d} m^{-1}[\operatorname{Cov}(\sum_{\gamma=1}^{m} Z_{\gamma i}, \sum_{\gamma=1}^{m} Z_{\gamma i})-\sum_{\gamma=1}^{m}\operatorname{Cov}(Z_{\gamma i}, Z_{\gamma i})]$$

$$=2\|\lambda\|^2 \sum_{i,j=1}^{d} [\operatorname{Cov}(m^{-1/2}\sum_{\gamma=1}^{m} k^{-1/2}\sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i}, m^{-1/2}\sum_{\gamma=1}^{m} k^{-1/2}\sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i}$$

$$-m^{-1}\sum_{\gamma=1}^{m}\operatorname{Cov}(k^{-1/2}\sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i}, k^{-1/2}\sum_{\mu=1}^{k} X_{k(\gamma-1)+\mu i})]$$

$$=2\|\lambda\|^2 \sum_{i,j=1}^{d} ((km)^{-1}\operatorname{Cov}(\sum_{\mu=1}^{km} X_{\mu i}, \sum_{\mu=1}^{km} X_{\mu i})-k^{-1}\operatorname{Cov}(\sum_{\mu=1}^{k} X_{\mu i}, \sum_{\mu=1}^{k} X_{\mu i}))$$

$$=2\|\lambda\|^2 \sum_{i,j=1}^{d} (\sigma_{ij}(km)-\sigma_{ij}(k))$$

$$\leq 2\|\lambda\|^2 \sum_{i,j=1}^{d} (A_{ij}-\sigma_{ij}(k))\to 0 \text{ as } k\to\infty. \quad (3.12)$$

Note that by the assumption (i) of this theorem the above convergence holds. Recall that $\sigma_{ij}(n)\leq A_{ij}$ and $\{\sigma_{ij}(n)\}$ is an increasing sequence. Finally, by $e^{ix}\simeq 1+ix-(x^2/2)$ and mean zero we have

$$E(\exp\{im^{-1/2}(k^{-1/2}\sum_{\mu=1}^{k}Y_\mu))\simeq 1-(1/2mk)E(\sum_{\mu=1}^{k}Y_\mu)^2$$

$$=1-(1/2m)E(\sum_{\mu=1}^{k}Y_\mu/\sqrt{k})^2$$

$$=1-\frac{1}{2m}E(\sum_{\mu=1}^{k}\sum_{i=1}^{d}X_{\mu i}\lambda_i/\sqrt{k})^2$$

$$=1-\frac{1}{2m}E(\sum_{i=1}^{d}\lambda_i S_{ki}/\sqrt{k})^2$$

$$=1-\frac{1}{2m}\sum_{i=1}^{d}\sum_{j=1}^{d}(\lambda_i\sigma_{ij}(k)\lambda_j)$$

$$=1-\frac{1}{2m}(\lambda\Sigma_k\lambda^t) \tag{3.13}$$

and

$$\exp\left(-\frac{1}{2}\lambda A\lambda^t\right)=\exp\left(-\frac{1}{2m}\lambda A\lambda^t\right)^m\simeq\left(1-\frac{1}{2m}\lambda A\lambda^t\right)^m. \tag{3.14}$$

Since $\sigma_{ij}(k)\to A_{ij}$ as $k\to\infty$ (assumption (i) of this theorem)

$$\sum_k\to A \text{ as } k\to\infty. \tag{3.15}$$

Applying (3.13), (3.14), and (3.15) to (III) we obtain

$$|\prod_1^m E\exp\{im^{-1/2}(k^{-1/2}\sum_{\mu=1}^{k}Y_\mu)\}-\exp\left(-\frac{1}{2}\lambda A\lambda^t\right)|\to 0 \text{ as } k\to\infty \tag{3.16}$$

By combining (3.4), (3.12), and (3.16) we complete the proof.

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