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MONOTONICITY OF PERMANENTS OF CERTAIN DOUBLY STOCHASTIC MATRICES

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A nonnegative matrix is called *doubly stochastic* if all row sums and column sums equal 1. The set of all $n \times n$ doubly stochastic matrices, denoted by Ω_n, forms a convex polytope with permutation matrices as vertices [7]. We denote by I_n the $n \times n$ identity matrix, and by K_n the $n \times n$ matrix all of whose entries equal 1. One face of Ω_n of special interest is the face $\Omega(R_n)$ of derangement matrix $R_n (=K_n - I_n)$. Let $\partial\Omega(R_n)$ denote the boundary of $\Omega(R_n)$. For this face $\Omega(R_n)$, we have a conjecture as follows;

"*The minimum permanent on $\Omega(R_n)$ is $d_n/(n-1)^n$, where d_n is the n-th derangement number, and that it occurs uniquely at $\frac{1}{n-1} R_n$*".

(Problem 3 in [3] and Conjecture 44 in [8])

In this paper, we study the stronger version than this conjecture, that is, the function

$$f_A(\lambda) = \text{per}(\lambda A + (1-\lambda) D_n)$$

is strictly increasing in the interval $0 \leq \lambda \leq 1$, where A is any fixed matrix on the boundary $\partial\Omega(R_n)$ of $\Omega(R_n)$ and $D_n = \frac{1}{n-1} R_n$ for $n \geq 3$.

PROPOSITION 1. *For all $A \in \Omega(R_3)$, $A \neq D_3$, $f_A(\lambda)$ is strictly increasing in the interval $0 \leq \lambda \leq 1$.*

Proof. For arbitrary $A = \begin{bmatrix} 0 & a & 1-a \\ 1-a & 0 & a \\ a & 1-a & 0 \end{bmatrix}$

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$\in\Omega(R_3)$, let $A_1=\lambda_1A+(1-\lambda_1)D_3$ and $A_2=\lambda_2A+(1-\lambda_2)D_3$, where $0\leqq\lambda_1<\lambda_2\leqq1$. Then per $A_1=3\lambda_1^2\left(a-\frac{1}{2}\right)^2+\frac{1}{4}$. Therefore,

$$\text{per } A_2-\text{per } A_1=3\left(a-\frac{1}{2}\right)^2(\lambda_2^2-\lambda_1^2)\geqq0,$$

where equality holds if and only if $A=D_3$.

LEMMA 2. ([4]) *If A is a nonnegative doubly stochastic matrix on $\Omega(R_n)$ sufficiently close to D_n and $A\neq D_n$, then per $A>$per D_n. In other words, the permanent function has a strict local minimum at D_n on $\Omega(R_n)$.*

THEOREM 3. *If $A\in\Omega(R_n)$ and $A\neq D_n$ then these are equivalent;*

(1) *$f_A(\lambda)$ is nondecreasing in the interval $0\leqq\lambda\leqq1$.*

(2) *$f_A(\lambda)$ is strictly increasing in the interval $0\leqq\lambda\leqq1$.*

Proof. Being a polynomial in $\lambda, f_A(\lambda)$ cannot be a constant over a subinterval of $[0,1]$ without being a constant throughout. Therefore, if $f_A(\lambda)=c$ over a subinterval $[\lambda_1,\lambda_2]$ of $[0,1]$, $\lambda_1<\lambda_2$, then $f_A(\lambda)=c$ over all $[0,1]$. However, this contradicts Lemma 2.

LEMMA 4. *Let $A\in\Omega(R_n)$ and $A\neq D_n$. Then there exists $A_0\in\partial\Omega(R_n)$ and $\lambda_0\in(0,1]$ such that $A=\lambda_0A_0+(1-\lambda_0)D_n$. Furthermore, if $f_{A_0}(\lambda)$ is nondecreasing, so is $f_A(\lambda)$.*

Proof. Let $L(A)$ be the line segment through D_n and A intersecting the boundary of $\Omega(R_n)$ at A_0. That is, if $a_{st}=\min\{a_{ij}|i\neq j\}$ and $\lambda_0=1-(n-1)a_{st}>0$, then $A_0=(1/\lambda_0)(A-(1-\lambda_0)D_n)$ is a doubly stochastic matrix with at least one zero except main diagonal, and

$$L(A)=\{S|S=\lambda A_0+(1-\lambda)D_n,\ \ 0\leqq\lambda\leqq1\}.$$

Clearly, $A=\lambda_0A_0+(1-\lambda_0)D_n\in L(A)$.

Furthermore, assume that $f_{A_0}(\lambda)=\text{per}(\lambda A_0+(1-\lambda)D_n)$ is nondecreasing for $0\leqq\lambda\leqq1$. Then we have

$$\begin{aligned}f_A(\lambda)&=\text{per}(\lambda A+(1-\lambda)D_n)\\&=\text{per}(\lambda(\lambda_0A_0+(1-\lambda_0)D_n)+(1-\lambda)D_n).\end{aligned}$$

Hence $f_A(\lambda)$ is also nondecreasing.

Now, we pose a conjecture on $\Omega(R_n)$ for $n\geqq4$.

CONJECTURE M(D) : *For all* $A \in \partial\Omega(R_n)$, $f_A(\lambda)=\mathrm{per}(\lambda A+(1-\lambda)D_n)$ *is nondecreasing in the interval* $0 \leqq \lambda \leqq 1$.

COROLLARY 5. *If we assume the conjecture* $M(D)$, *then* $f_A(\lambda)$ *is nondecreasing for all* $A \in \Omega(R_n)$.

Proof. Assume that the conjecture M(D) holds. By Lemma 4, for arbitrary $A \in \Omega(R_n)$ and $A \neq D_n$, there is $A_0 \in \partial\Omega(R_n)$ and $\lambda_0 \in (0, 1]$ such that $A=\lambda_0 A_0+(1-\lambda_0)D_n$. Since $f_{A_0}(\lambda)$ is nondecreasing by assumption, we have $f_A(\lambda)$ is also nondecreasing by Lemma 4.

COROLLARY 6. *If we assume the conjecture* $M(D)$ *for* $n \geqq 3$, *then we have the followings;*

(i) D_n *is the minimizing matrix on* $\Omega(R_n)$,

(ii) *For all* $A \in \Omega(R_n)$, $f_A'(1) \geqq 0$,

(iii) *For* $A \in \Omega(R_n)$, *define* $\phi(A)=(R_n-A)/(n-2)$ *and in general,* $\phi^{(m+1)}(A)=\phi(\phi^{(m)}(A))$ *by iteration. Then if* $A \in \Omega(R_n)$ *and* $A \neq D_n$, per $A >\mathrm{per}(\phi^{(2k)}(A))$, $k=1, 2, \cdots$,

(iv) *For* $A \in \Omega(R_n)$, *define* $\psi(A)=(R_n+A)/n$ *and in general,* $\psi^{(m+1)}(A)=\psi(\psi^{(m)}(A))$ *by iteration. Then if* $A \in \Omega(R_n)$ *and* $A \neq D_n$, per $A>$ per $(\psi(A))$.

Proof. (i) Using Corollary 5 and Theorem 3, we have $f_A(\lambda)$ is strictly increasing in interval $0 \leqq \lambda \leqq 1$. Hence D_n is the minimizing matrix on $\Omega(R_n)$.

(ii) By Corollary 5, we have $f_A'(\lambda) \geqq 0$ for $\lambda \in (0, 1)$. But $f_A'(\lambda)$ is a polynomial in λ and hence $f_A'(1) \geqq 0$.

(iii) Now, we compute

$$\begin{aligned}\phi(A)&=(R_n-A)/(n-2)\\&=\left(1+\frac{1}{n-2}\right)D_n+\left(-\frac{1}{n-2}\right)A\end{aligned}$$

and

$$\phi^2(A)=\left(1-\frac{1}{(n-2)^2}\right)D_n+\frac{1}{(n-2)^2}A.$$

Since $A \neq D_n$, $f_A(\lambda)$ is strictly increasing in the interval $0 \leqq \lambda \leqq 1$ by Corollary 5 and Theorem 3. Hence

$$\text{per } A=f_A(1)>f_A\left(\frac{1}{(n-2)^2}\right)=\text{per}(\phi^{(2)}(A)).$$

Similarly, we have per $A>\text{per}(\phi^{(2k)}(A))$, for $k=1,2,\cdots$.

(iv) Similar to the proof of (iii), using

$$\phi(A)=\frac{1}{n}A+\left(1-\frac{1}{n}\right)D_n.$$

We remark that (i) is the Conjecture 44 in [8].

LEMMA 7. ([2]) *For $A=(a_{ij})\in\Omega_n$, let $\tau:\Omega_n\to\Omega_n$ be the transformation defined by*

$$(\tau(A))_{ij}=\{a_{ij}\text{ per } A(i|j)\}/\text{per } A.$$

Then per $A\leq\text{per}(\lambda A+(1-\lambda)\tau(A))$ *for* $\lambda\in[0,1)$ *and the equality holds if and only if* $\tau(A)=A$.

LEMMA 8. *Let $A\in\partial\Omega(R_n)$. Suppose that, for $\lambda\in(0,1)$, there exists λ' such that $\lambda<\lambda'$ and*

$$\tau(\lambda A+(1-\lambda)D_n)=\lambda' A+(1-\lambda')D_n.$$

Then $f_A(\lambda)$ is strictly increasing in the interval $0\leq\lambda\leq1$.

Proof. Since $A\in\partial\Omega(R_n)$, there exists $a_{ij}=0$ for some $i\neq j$. Then $(\lambda A+(1-\lambda)D_n)_{ij}=(1-\lambda)d_{ij}$ and $(\lambda' A+(1-\lambda')D_n)_{ij}=(1-\lambda')d_{ij}$ are different each other because $\lambda<\lambda'$. Hence $\lambda A+(1-\lambda)D_n$ and $\lambda' A+(1-\lambda')D_n$ are different. By Lemma 7,

$$\begin{aligned}\text{per}(\lambda A+(1-\lambda)D_n) &< \text{per}(\tau(\lambda A+(1-\lambda)D_n))\\ &= \text{per}(\lambda' A+(1-\lambda')D_n).\end{aligned}$$

Hence $\text{per}(\lambda A+(1-\lambda)D_n)$ is strictly increasing in the interval $0\leq\lambda\leq1$.

Assume $1<s,t$. Let X denote an s-square matrix with variables $x_{ij}=x(i\neq j)$, $x_{ii}=0$, Y an $s\times t$ matrix with all entries equal to a variable y, and Z a $t\times t$ matrix with variables $z_{ij}=z(i\neq j)$, $z_{ii}=0$. Let

$$L=L(x,y,z)=\begin{bmatrix}X & Y\\ Y^t & Z\end{bmatrix}.$$

Note that, for nonnegative x,y and z, $L\in\Omega(R_{s+t})$ if and only if

$(s-1)x+ty=sy+(t-1)z=1$. We define $f_{s,t}=\text{per } L$, $g_{s,t}=\text{per } L(1|2)$, $h_{s,t}=\text{per } L(1|s+t)$. Then we have

$$f_{s,t}=(s-1)x\cdot g_{s,t}+ty\cdot h_{s,t} \qquad (*)$$

by expanding $f_{s,t}$ along the first row of L.

LEMMA 9. *For all x, y and z such that* per $L\neq 0$, *there exist x', y' and z' such that $\tau(L(x,y,z))=L(x',y',z')$.*

Proof. It suffices to note that the $(s+t-1)\times(s+t-1)$ submatrix of an entry in the block Y and that of the corresponding entry in the block Y^t are transposes of each other and hence have the same permanent.

LEMMA 10. *Let $L(x,y,z)\in\Omega(R_{s+t})$, $1<s,t$. Then there exist λ such that $L(x,y,z)=\lambda(D_s\oplus D_t)+(1-\lambda)D_{s+t}$.*

Proof. Let $n=s+t$ and $\lambda=1-(n-1)y$. Since $L(x,y,z)\in\Omega(R_n)$ and $(s-1)x+ty=sy+(t-1)z=1$, we have $(s-1)(x-y)=(1-ty)-(s-1)y=1-(s+t-1)y=\lambda$ and $(t-1)(z-y)=(1-sy)-(t-1)y=1-(s+t-1)y=\lambda$. Therefore,

$$\lambda(D_s\oplus D_t)+(1-\lambda)D_n=L\left(\frac{\lambda}{s-1}+\frac{1-\lambda}{n-1},\ \frac{1-\lambda}{n-1},\ \frac{\lambda}{t-1}+\frac{1-\lambda}{n-1}\right)$$
$$=L(x,y,z).$$

THEOREM 11. *Let $A=D_s\oplus D_t\in\Omega(R_n)$ and $1<s,t$. If we assume that $g_{s,t}>h_{s,t}$ for $L(x,y,z)=\lambda A+(1-\lambda)D_{s+t}$, then $f_A(\lambda)=\text{per}(\lambda A+(1-\lambda)D_{s+t})$ is strictly increasing in the interval $0\leqq\lambda\leqq 1$.*

Proof. Assume $s\geqq t$ without loss of generality. Let $n=s+t$ and $\lambda\in(0,1]$. Then it is readily verified that

$$\lambda(D_s\oplus D_t)+(1-\lambda)D_{s+t}=L(x,y,z),$$

where $x=\dfrac{(s-1)+t\lambda}{(s-1)(n-1)}$, $y=\dfrac{1-\lambda}{n-1}$, $z=\dfrac{(t-1)+s\lambda}{(t-1)(n-1)}$. By Lemma 9, there exist x', y' and z' such that $\tau(L(x,y,z))=L(x',y',z')$. By Lemma 10, there is λ' such that $L(x',y',z')=\lambda'(D_s\oplus D_t)+(1-\lambda')D_{s+t}$. That is, $\tau(\lambda(D_s\oplus D_t)+(1-\lambda)D_n)=\lambda'(D_s\oplus D_t)+(1-\lambda')D_n$. Our result would follow from Lemma 8 if $\lambda<\lambda'$. Since λ' satisfies the equality

$(\tau(L))_{1,2}=(L)_{1,2}\times$per $L(1|2)/$per L from Lemma 7, we have $\lambda'/(s-1)+(1-\lambda')/(n-1)=\{\lambda/(s-1)+(1-\lambda)/(n-1)\}\times(g_{s,t}/f_{s,t})$, that is

$$\frac{(s-1)+t\lambda'}{(s-1)(n-1)}=\frac{(s-1)+t\lambda}{(s-1)(n-1)}\times\frac{g_{s,t}}{f_{s,t}}. \qquad (**)$$

Hence it suffices to show that $g_{s,t}>f_{s,t}$. Since $(s-1)x+ty=1$, we get by (*) that

$$\begin{aligned}g_{s,t}-f_{s,t}&=(1-(s-1)x)\cdot g_{s,t}-ty\cdot h_{s,t}\\&=ty\cdot(g_{s,t}-h_{s,t})>0\end{aligned}$$

by assumption.

THEOREM 12. *Let* $A=L\left(0,\ \frac{1}{t},\ \frac{t-s}{t(t-1)}\right)$, *where* $t\geqq s\geqq 1$. *If we assume* $g_{s,t}<h_{s,t}$, *then* per$(\theta A+(1-\theta)D_{s+t})$ *is strictly increasing in the interval* $0\leqq\theta\leqq 1$.

Proof. Since $A=L\left(0,\ \frac{1}{t},\ \frac{t-s}{t(t-1)}\right)=\left(-\frac{s-1}{t}\right)\times(D_s\oplus D_t)+\left(1+\frac{s-1}{t}\right)D_{s+t}$, $\theta A+(1-\theta)D_{s+t}=\left(-\frac{(s-1)\theta}{t}\right)\times(D_s\oplus D_t)+\left(1+\frac{\theta(s-1)}{t}\right)D_{s+t}$. It remains to show that $f_B(\lambda)$ is strictly decreasing in the interval $-\frac{s-1}{t}\leqq\lambda<0$, where $B=D_s\oplus D_t$. By Lemma 9 and 10, there exists λ' such that $\tau(\lambda(D_s\oplus D_t)+(1-\lambda)D_n)=\lambda'(D_s\oplus D_t)+(1-\lambda')D_n$. Our result would follow from Lemma 8 if $\lambda>\lambda'$. Since λ' satisfies the equality $\tau(A)_{12}=\{a_{12}$ per $A(1|2)\}/$per A from Lemma 7, we have the equation (**). Hence it suffices to show that $g_{s,t}<f_{s,t}$. Since $(s-1)x+ty=1$, we get by (*) that

$$\begin{aligned}g_{s,t}-f_{s,t}&=(1-(s-1)x)g_{s,t}-ty\cdot h_{s,t}\\&=ty\cdot(g_{s,t}-h_{s,t})<0\end{aligned}$$

by assumption.

EXAMPLES 13. (1) For the condition $g_{s,t}>h_{s,t}$ in Theorem 11, we consider the matrix

$$L(x,y,z)=\begin{bmatrix}X & Y\\ Y^t & Z\end{bmatrix}\in\Omega(R_{3+2}).$$

Then

$$g_{3,2}-h_{3,2}=2xy^2(z-y)+(2y^2+xz)(2y^2+xz-3xy)$$
$$=\frac{\lambda}{32}(6\lambda^3-\lambda^2+2\lambda+1)>0$$

for $x=\frac{1+\lambda}{4}$, $y=\frac{1-\lambda}{4}$, $z=\frac{1+3\lambda}{4}$ and $\lambda\in(0,1]$. Hence the condition $g_{s,t}>h_{s,t}$ in Theorem 11 holds for the matrix $L(x,y,z)\in\Omega(R_{3+2})$.

(2) For the condition $g_{s,t}<h_{s,t}$ in Theorem 12, we consider the matrix

$$L\left(0,\ \frac{1}{t},\ \frac{t-s}{t(t-1)}\right)=\begin{bmatrix} 0 & Y \\ Y^t & Z \end{bmatrix}\in\Omega(R_{s+t}).$$

(1) If $s=t>1$, then we have that

$$g_{s,t}=0<\frac{t!}{t^t}\times\frac{(t-1)!}{t^{t-1}}=h_{s,t}.$$

(2) If $s=t-1>1$, then we have that

$$g_{s,t}=\left[\frac{(t-2)!}{t^{t-2}}\right]^2\times\frac{t^2-3t+3}{2t(t-1)} \quad \text{and}$$
$$h_{s,t}=\left[\frac{(t-1)!}{t^{t-1}}\right]^2.$$

Hence we have that

$$h_{s,t}-g_{s,t}=\left[\frac{(t-2)!}{t^{t-1}}\right]^2\times\left[\frac{(t-1)^3-1}{2(t-1)}\right]>0$$

for all $t>2$.

Therefore, the assumption $g_{s,t}<h_{s,t}$ holds for the matrices $L\left(0,\ \frac{1}{t},\ \frac{t-s}{t(t-1)}\right)\in\Omega(R_{s+t})$ whenever $s=t$ or $s=t-1$.

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