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REGULAR NEAR-RING MODULES

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Introduction

By analogy with the concept of a regular module introduced by Zelmanowitz[6], in this paper we introduce the concept of a regular near-ring module. If N is a near-ring then an N-module M is called regular if for each $m\in M$ there exists $f\in \mathrm{Hom}_N(M, N)$ such that $m=(fm)m$. It follows that any regular near-ring may be regarded as a regular N-module.

The purpose of this paper is to investigate some properties of regular near-ring modules and to characterize regularity of both the endomorphism semigroup $\mathrm{End}_N(M)$ of a regular N-module M and the center of $\mathrm{End}_N(M)$. Finally, we show that if R is a ring with identity and M is a unital R-module, then the corresponding centralizer near-ring is strictly semiprime. The most results in this paper are to generalize the corresponding ones in [6].

Throughout this paper, N stand for a right near-ring in which $n0=0$ for each $n\in N$. For the basic terminology and notation we refer to Pilz [4].

THEOREM 1. *A cyclic regular N-module is isomorphic to an N-subgroup of N generated by an idempotent.*

Proof. If m is a generator of a cyclic regular N-module M, then $m=(fm)m\in Nm$ for some $f\in \mathrm{Hom}_N(M, N)$. It follows that $M=Nm$. Suppose that $f(nm)=f(n_1m)$ for $n, n_1\in N$. Then $nm=n((fm)m)=f(nm)m=f(n_1m)m=n_1(f(m)m)=n_1m$. Therefore f is monomorphism. Since $fM=f(Nm)=N(fm)$ and fm is an idempotent element of N, we

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have $M \cong N(fm)$.

An N-module M is called the semi-direct sum of its N-subgroups A and B (denoted by $M = A \dot{+} B$) if A is a submodule of M, $M = A + B$ and $A \cap B = \{0\}$. Here A is called a semi-direct summand of M.

LEMMA 2. *Let M be an N-module, $f \in \mathrm{Hom}_N(M, N)$ and $m \in M$. Then there exists $f_m \in \mathrm{End}_N(M)$ such that $f_m(m') = (fm')m$ for each $m' \in M$.*

Proof. It is obvious.

THEOREM 3. *Let M be a regular N-module and A a cyclic N-subgroup of M. Then there exists a submodule B of M such that $M = A \dot{+} B$.*

Proof. Let A be a cyclic N-subgroup of M. Then there exists $m \in M$ such that $A = Nm$. Choose $f \in \mathrm{Hom}_N(M, N)$ with $m = (fm)m$. Let m' be any element of M. Then, by Lemma 2, there is $f_m \in \mathrm{End}_N(M)$ such that $f_m(m') = (fm')m$. Since f_m is an idempotent in $\mathrm{End}_N(M)$, $m' = f_m m' + (m' - f_m m') \in Nm + \mathrm{Ker}\ f_m$. It is easy to show that $Nm \cap \mathrm{Ker}\ f_m = \{0\}$. Hence $M = A \dot{+} \mathrm{Ker}\ f_m$.

THEOREM 4. *Let M be an N-module. Then for each $f \in \mathrm{End}_N(M)$, the following statements are equivalent.*

(a) *There exists $g \in \mathrm{End}_N(M)$ such that $fgf = f$.*

(b) *There exists an N-subgroup M_1 and an N-submodule M_2 such that $\mathrm{Ker}\ f \dot{+} M_1 = M$ and $\mathrm{Im}\ f \dot{+} M_2 = M$.*

Proof. (a)$\Rightarrow$(b). Obvious.

(b)$\Rightarrow$(a). Assume (b) holds. Then there exists $g' : \mathrm{Im}\ f \to M$ such that $fg'y = y$ for all $y \in \mathrm{Im}\ f$, that is, $fg'fx = fx$ for all $x \in M$. But $M = \mathrm{Im}\ f \dot{+} M_2$ for some N-submodule M_2, so we can extend g' on M by taking $g = 0$ on the complementary of M_2. Hence for any $x \in M$, $fgfx = fx$. Thus $fgf = f$.

REMARK 5. $\mathrm{End}_N(M)$ is not always a near-ring under pointwise addition and composition. But it is a semigroup under the composition. Hence we obtained a characterization of regularity of the endomorphism semigroup $\mathrm{End}_N(M)$.

THEOREM 6. *Let M be an N-module and let $\alpha\in$*Center *of* $\mathrm{End}_N(M)$. *Then the following statements are equivalent.*

(a) *There exists $\beta\in$*Center *of* $\mathrm{End}_N(M)$ *such that $\alpha\beta\alpha=\alpha$.*

(b) $M=\alpha M\dot{+}\mathrm{Ker}\ \alpha$.

Proof. (a)$\Rightarrow$(b). Suppose that such α,β exist. Let $\eta=\alpha\beta=\beta\alpha$. Then $\eta^2=\eta$ and $\alpha M=\eta M$. It follows that Ker $\alpha=$Ker η. Hence we have $M=\eta M\dot{+}\mathrm{Ker}\ \eta$.

(b)$\Rightarrow$(a). Suppose that $\alpha\in$Center of $\mathrm{End}_N(M)$ and $M=\alpha M\dot{+}\mathrm{Ker}\ \alpha$. Given $m\in M$ write $m=\alpha n+k$ with $n\in M$ and $k\in$Ker α, and in turn write $n=\alpha n_1+k_1$ with $n_1\in M$ and $k_1\in$Ker α. Then $\alpha m=\alpha^2(\alpha n_1)$. Let $x_m=\alpha n_1$. It follows that x_m is the unique element of αM such that $\alpha m=\alpha^2x_m$. We can easily show that $x_{nm}=nx_m$, $x_{m+m_1}=x_m+x_{m_1}$, $x_{rm}=rx_m$ for any $m, m_1\in M$, $n\in N$ and $r\in\mathrm{End}_N(M)$.

Define $\beta\in\mathrm{End}_N(M)$ by $\beta m=x_m$ for each $m\in M$. It follows that $\alpha\beta\alpha=\alpha$ and $\beta\in$Center of $\mathrm{End}_N(M)$.

THEOREM 7. *Let M be a regular N-module. Then the* Center *of* $\mathrm{End}_N(M)$ *is a regular semigroup.*

Proof. Let $\alpha\in$Center of $\mathrm{End}_N(M)$. For any $m\in M$, choose $f\in\mathrm{Hom}_N(M,N)$ such that $\alpha m=(f(\alpha m))\alpha m$. Since $\alpha m=\alpha^2f_mm$, we have $m=\alpha(f_mm)+(m-\alpha(f_mm))$. Hence $M=\alpha M+\mathrm{Ker}\ \alpha$.

Moreover, $\alpha M\cap\mathrm{Ker}\ \alpha=0$. Thus $M=\alpha M\dot{+}\mathrm{Ker}\ \alpha$. It follows from Theorem 6 that $\mathrm{End}_N(M)$ is a regular semigroup.

Due to Oswald[3], we say that a near-ring N is strictly semiprime if $A^2=\{0\}$ implies $A=\{0\}$ where A is an N-subgroup of N.

LEMMA 8 ([3]). *If N has the property that $xNx=\{0\}$ implies $x=0$, then N is strictly semiprime.*

Let R be a ring with identity and M a unital R-module. The corresponding centralizer near-ring is $C(R;M)=\{f: M\to M\mid f(rm)=r(fm)$ for all $r\in R$, $m\in M\}$ ([2]).

THEOREM 9. *Let R be a ring with identity and ${}_RM$ a regular R-module. Then $C(R;M)$ is strictly semiprime.*

Proof. Let $\alpha(\neq 0)\in C(R;M)$. Then there exists $m\in M$ such that

$\alpha m \neq 0$. Since M is regular, we have an $f \in \mathrm{Hom}_R(M, R)$ with $\alpha m = (f(\alpha m))\alpha m$. It follows that $\alpha m = (\alpha f_m \alpha) m$. Hence $0 \neq \alpha f_m \alpha \in \alpha C(R;M)\alpha$. It follows from Lemma 9 that $C(R;M)$ is strictly semiprime.

COROLLARY 10. *Let R be a ring with identity and ${}_RM$ a regular R-module. Then $C(R;M)$ has no nonzero nilpotent N-subgroup.*

Proof. Combining Theorem 10 and Theorem 1 of [3], it is immediate.

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