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ON PREORDERINGS OF HIGHER LEVEL

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1. Introduction

Lam investigated some properties on Krull valuation which is compatible with a preordering of a field [9]. Furthermore he showed many results on preorderings and its related topics on the Henselization (F_1, v_1) of a valuated field (F, v). Especially the concept of preordering was developed to higher level one and many results which held in the case of preordering were generalized to those of higher level preordering by E. Becker [4]. We continually proceed to find some properties that are related with higher level preordering.

2. Preliminaries

A subset P of a field F is called a preordering of level $2n$ (n is a natural number, char $F=0$) if it satisfies the following conditions [2].

(1) $P+P\subset P$ (2) $P\cdot P\subset P$

(3) $F^{2n}\subset P$ (4) $-1\notin P$

For any abelian group G, we write $G^*=\mathrm{Hom}(G, \mu)$ where $\mu=\{\zeta\in \boldsymbol{C} \mid \zeta^r=1$ for some $r\in \boldsymbol{N}\}$. Let $\dot{F}=F-\{0\}$. $\chi\in\dot{F}^*$ is called a signature of F if Ker χ is additively closed [3, 4]. The subset of signatures of $\dot{F}^*$ is denoted by $\mathrm{SGN}(F)$.

If $v: F\to\Gamma$ is a Krull valuation with the ordered abelian group written multiplicatively, we denote the valuation ring of v by A, the maximal ideal of A by $\mathscr{M}$, the group of units of A by U. we shall often write $(v, A, \mathscr{M}, \Gamma, \cdots)$. For a valuated field (F, v), $\bar{T}$ denotes the images of $T\cap A$ in residue class field $\bar{F}=A/\mathscr{M}$. $\chi\in\mathrm{SGN}(F)$ is said to be compatible with a valuation, written $v\sim\chi$, if $(1+\mathscr{M})\subset\mathrm{Ker}\ \chi$. Let T be a preordering of level $2n$ and $X_T=\{\chi\in\mathrm{SGN}(F) : \chi|_{\dot{T}}=1\}$. The set of valuations which are compatible with χ for some $\chi\in X_T$ is denoted by

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$\Omega(T)$ [4].

3. Preorderings of Higher Level on an Henselian Field

DEFINITION 3.1. Let T be a preordering of level $2n$ and v be a Krull valuation on a field F. $v\in\Omega^*(T)$ means $v\sim\chi$ for all $\chi\in\mathrm{SGN}(F)$ satisfying $\chi(\dot{T})=1$ ($v\sim\chi$ for all $\chi\in X_T$).

PROPOSITION 3.2. *Let $(v, A, \mathscr{M}, \Gamma, \cdots)$ be a valuation on a field F and T be a preordering of level $2n$. Then*

(1) *$v\in\Omega^*(T)$ if and only if $1+\mathscr{M}\subset\dot{T}$*

(2) *$v\in\Omega(T)$ if and only if $\bar{T}$ is a preordering of level $2n$ if and only if $(1+\mathscr{M})\cap(-T)=\phi$.*

Proof. (1) By Becker [4], $\dot{T}=\bigcap_{\chi\in X_T}\mathrm{Ker}(\chi)$ where T is a preordering of level $2n$. $v\in\Omega^*(T)$ iff $v\sim\chi$ for any $\chi\in X_T$ iff $1+\mathscr{M}\subset\mathrm{Ker}(\chi)$ for any $\chi\in X_T$ iff $1+\mathscr{M}\subset\bigcap_{\chi\in X_T}\mathrm{Ker}(\chi)=\dot{T}$.

(2) Suppose $v\in\Omega(T)$. By Becker [4], $\bar{T}$ is a preordering of level $2n$. Suppose $\bar{T}$ is a preordering of level $2n$. Then we have $-\bar{1}\notin\bar{T}$. Hence $(-T)\cap(1+\mathscr{M})=\phi$. Otherwise $1+m=-t$ implies $t=-1-m\in A$ and $-\bar{1}=\bar{t}\in\bar{T}$, it is absurd. Suppose $(-T)\cap(1+\mathscr{M})=\phi$. Then clearly $-\bar{1}\notin\bar{T}$ and $\bar{T}$ is a preordering of level $2n$: Let $\bar{t}_1, \bar{t}_2\in\dot{\bar{T}}$. Then $t_1, t_2\in T\cap U\subset A$. Assume $t_1+t_2\in\mathscr{M}$. Since $t_1+t_2=m$, $t_1=-t_2+m=t_2(-1+m/t_2)$, $t_1/t_2=-1+m/t_2\in A$ and $t_1/t_2=-1+t_2^{-1}m\in-1+\mathscr{M}\subset A$ so $-\bar{1}\in\bar{T}$. It is absurd. Hence $t_1+t_2\subset U\cap T\subset A$ and $\bar{t}_1+\bar{t}_2\in\dot{\bar{T}}$. Let $\bar{F}^{2n}\cdot\bar{T}\subset\overline{F^{2n}T}\subset\bar{T}$ and $\dot{\bar{T}}$ is closed under multiplication. So we have $\bar{T}$ is a preordering of level $2n$. Suppose $\bar{T}$ is a preordering of level $2n$. Since T is a preordering of level $2n$ by hypothesis, we have $v\in\Omega(T)$ by Becker [4]. Therefore we proved the proposition.

Let v be a valuation on a field F, T be a preordering of level $2n$ and Q be a preordering of level $2n$ containing $\bar{T}$. Then $T\wedge Q=T\cdot\pi^{-1}(Q)$ is a preordering of level $2n$ on F with $\overline{T\wedge Q}=Q$ [9] (π is the projection map $A\to A/\mathscr{M}$).

PROPOSITION 3.3. *Let v be as above. Then $v\in\Omega^*(T\wedge Q)$.*

Proof. Since $T\wedge Q=\bigcap_{\chi\in X_{T\wedge Q}}\mathrm{Ker}(\chi)$ and $\overline{1+\mathscr{M}}=\bar{1}\in\dot{Q}$, we have $\bigcap_{\chi\in X_{T\wedge Q}}$

$\text{Ker}(\chi)=T\wedge Q=T\cdot\pi^{-1}(\dot{Q})\supset\pi^{-1}(\dot{Q})\supset 1+\mathcal{M}$. Then $1+\mathcal{M}\subset\text{Ker}(\chi)$ for any $\chi\in X_{T\wedge Q}$. Therefore $\chi\sim v$ for any $\chi\in X_{T\wedge Q}$ i.e. $v\in\Omega^*(T\wedge Q)$.

A field F with a valuation $(v, A, \mathcal{M}, \bar{F}, \cdots)$ is said to be Henselian if Hensel's Lemma holds over the valuation ring A. Let (F_1, v_1) be an extension of valuated field (F, v), i.e. $F_1\supset F, v_1(\dot{F}_1)\supset v(\dot{F})$ and $v_1|_F=v$. We say that (F_1, v_1) is an immediate extension of (F, v) if $v_1(\dot{F}_1)=v(\dot{F})$ and $\bar{F}_1=\bar{F}$. To every valued field (F, v), there exists an immediate algebraic extension (F_1, v_1) which is Henselian [5]. The extension (F_1, v_1) is called a Henselization of (F, v).

PROPOSITION 3.4. *Let $(v, A, \mathcal{M}, \bar{F}, \cdots)$ be a valuation on F such that char $(\bar{F})=0$. If (F, v) is Henselian, then $1+\mathcal{M}\subset F^{2n}$ for any n.*

Proof. Let $m\in\mathcal{M}$ and consider $f(x)=x^{2n}-(1+m)\in A[x]$. Going to the residue field, $\bar{f}(x)=x^{2n}-\bar{1}=(x^n-\bar{1})(x^n+\bar{1})\in\bar{F}[x]$. Since char $(\bar{F})=0$ we have char $(\bar{F})\neq 2$ or char $(\bar{F})\neq n$, we have $\bar{1}$ is a simple root of $\bar{f}(x)$ on $\bar{F}$. Therefore $f(x)$ must have a root in A. i.e. $1+m\subset A^{2n}\subset F^{2n}$.

THEOREM 3.5. *Let (F, v) be an Henselian field. Suppose F is formally real. Then $v\in\Omega^*(T)$ for any preordering T of higher level $2n$.*

Proof. Let T be a preordering of level $2n$ in F. Assume char $(\bar{F})=k$ for some $k\neq 0$. Then $f(x)=x^2+x+k$ pushed down to $\bar{f}(x)=x^2+x=x(x+\bar{1})$. So there exists $c\in F$ such that $0=c^2+c+k=(c+1/2)^2+4(k-1)(1/2)^2+3(1/2)^2$. Since F is formally real. It is absurd. Hence char$(\bar{F})=0$. Then $1+\mathcal{M}\subset F^{2n}\subset T=\bigcap_{\chi\in X_T}\text{Ker}(\chi)$ by proposition 3.4, i.e. $v\in\Omega^*(T)$.

4. *T*-Forms

Let T be a preordering (of level $2n$). A T-form ρ of dimension r is an r-tuple $\langle a_1\dot{T}, \cdots, a_r\dot{T}\rangle$, $a_i\in\dot{F}$. We often write $\rho=\langle a_1, \cdots, a_r\rangle$, A T-form $\rho=\langle a_1, \cdots, a_r\rangle$ is T-isotropic if there exists $t_1, \cdots, t_r\in T$, not all 0, so that $\Sigma_1^r a_i t_i=0$; otherwise it is called T-anisotropic. If $v\in\Omega(T)$ and $a\in\dot{F}$, select those entries a_i in ρ with $v(a_i)\equiv v(a)$ mod $v(\dot{T})$. Set $\Gamma/(v(\dot{T}))=G$ and let $v(a)v(\dot{T})=g\in G$. Then

$$\rho=\perp_{g\in G}\langle a_{g,1}, \cdots, a_{g,r(g)}\rangle,$$

where $v(a_{g,i})v(\dot{T})=g$ and every entry a_j with $v(a_j)\cdot v(\dot{T})=g$ is an $a_{g,i}$. Put $\rho_a=\langle a^{-1}a_{g,i}, \cdots, a^{-1}a_{g,r(g)}\rangle$ [4].

PROPOSITION 4.1. *Suppose $v\in\Omega(T)$. If a T-form ρ diagonalized as $\rho=\perp_{g\in G}\langle a_{g,1}, \cdots, a_{g,r(g)}\rangle$ is T-isotropic, then at least one residue class form $\bar{\rho}_a$ is $\bar{T}$-isotropic.*

Proof. Suppose $\Sigma_{g\in G}\Sigma_i a_{g,i}t_{g,i}=0$ where $t_{g,i}\in T$ are not all zero. Among the nonzero summands in this equation, say $a_{h,1}t_{h,1}$ is such that $a_{g,i}t_{g,i}/a_{h,1}t_{h,1}\in A$ for all g and all i. Divide the equation by $a_{h,1}t_{h,1}$ and project from A to $A/\mathscr{M}=\bar{F}$. If $g\neq h\in G$, then $a_{g,i}t_{g,i}/a_{h,1}t_{h,1}\in\mathscr{M}$ since otherwise we would have the contradiction that $v(a_{g,i})\equiv v(a_{h,1})\ (\text{mod}\ v(\dot{T}))$. Because $UT/T=U/U\cap T\to\dot{\bar{F}}/\dot{\bar{T}}$, we have an equation $\overline{\Sigma a_{h,i}t_{h,i}/a_{h,1}t_{h,1}}=0\in\bar{F}$ where $a_{h,i}t_{h,i}/a_{h,1}t_{h,1}=ut$. Hence above assertion holds. [cf. 9, Theorem 4.5].

THEOREM 4.2. *Assume $v\in\Omega^*(T)$. Then a T-form ρ is T-isotropic iff at least one residue class form $\bar{\rho}_a$ is $\bar{T}$-isotropic.*

Proof. For the necessary part, $v\in\Omega^*(T)$ always implies $v\in\Omega(T)$. Hence by above proposition, it is clear. For the sufficient part, let $\rho=\perp_{g\in G}\langle a_{h,1}, a_{h,2}, \cdots, a_{h,n}\rangle$ satisfying above condition. Since $v(a_{h,i}/a_{h,1})\in v(\dot{T})$, we can write $a_{h,i}/a_{h,1}=t_i^{-1}u_i$ where $t_i\in\dot{T}$ and $u_i\in U$. Then $\overline{a_{h,i}/a_{h,1}}\in\bar{u}_i\dot{\bar{T}}$ and so by assumption, $\langle\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{n(h)}\rangle$ is $\bar{T}$-isotropic. Following the proof of [9, Theorem 4.6.], we can have above result by using the map $UT/U=U/U\cap T\to\dot{\bar{F}}/\dot{\bar{T}}$ and proposition 3.1.

LEMMA 4.3. *Let $(v, A, \mathscr{M}, \cdots)$ be a real valuation on F and $T=\Sigma F^{2n}$ for any n. Then $\bar{T}=\Sigma\bar{F}^{2n}$ (v is a real valuation if its residue field $\bar{F}=A/\mathscr{M}$ is a formally real field).*

Proof. $\bar{T}\supset\Sigma\bar{F}^{2n}$ is clear. We must show that $\bar{T}\subset\Sigma\bar{F}^{2n}$. Consider a nonzero element $a\in\dot{T}\cap A$ with $a=a_1^{2n}+a_2^{2n}+\cdots+a_r^{2n}$ for $a_i\in\dot{F}$. Say $v(a_1)=\min\{v(a_i)\}$. Since $v(a_i/a_1)=v(a_i)-v(a_1)\geq 0$, we have $a_i/a_1\in A$ for all i and $a=a_1^{2n}+\cdots+a_r^{2n}=a_1^{2n}(1+(a_2/a_1)^{2n}+\cdots+(a_r/a_1)^{2n})$. Since $\bar{F}$ is formally real, $\bar{F}$ is formally $2n$-real. Thus $1+(a_2/a_1)^{2n}+\cdots+(a_r/a_1)^{2n}$ is not contained in $\mathscr{M}$. Therefore it is a unit in A and we get $a\neq 0$, $v(a)=v(a_1^{2n})=2n\cdot v(a_1)=2n\min\{v(a_i)\}$. Hence $2n\cdot v(a_i)\geq v(a)$

≥ 0 for any i and so $a_i \in A$. Then $\bar{T}=\Sigma\bar{F}^{2n}$.

THEOREM 4.4. *Let* $(v, A, \mathscr{M}, \Gamma, \cdots)$ *be a real valuation on* F *and* $T=\Sigma F^{2n}$. *Let* $T^v = T\wedge\bar{T} = T(1+\mathscr{M})$ *and* (F_1, v_1) *be an Henselization of* (F, v) *and* $T_1=\Sigma F_1^{2n}$. *Then* (1) *A form* ρ *is* T^v*-isotropic iff* ρ *is weakly isotropic* (*i.e.* $n\rho$ *is isotropic for some* n). (2) $T^v = F\cap T_1$.

Proof. (1) Since (F_1, v_1) is a Henselization of (F, v), (F_1, v_1) is an immediate extension of (F, v), i.e. v_1 has the same value group Γ as v and $\bar{F}_1=\bar{F}$, with respect to the valuations v, v_1. Since v is a real valuation, $\bar{F}$ contains an ordering so $\bar{F}_1$ contains an ordering, then v_1 is a real valuation. Since $\bar{T}_1=\overline{\Sigma F_1^{2n}}=\Sigma\bar{F}_1^{2n}=\Sigma\bar{F}^{2n}=\overline{\Sigma F^{2n}}=\overline{T^v}(=\overline{T\wedge\bar{T}}=\bar{T})$ [4], we have $\bar{T}_1=\bar{T}$. Also since $T^v=T(1+\mathscr{M})$[4], $v(\dot{T}^v)=v(\dot{T}(1+\mathscr{M}))=v(\dot{T})=2n\Gamma=v_1(\dot{T}_1)$. Thus viewied as a T^v-form or as a T_1-form ρ has the same residue form after making suitable identification. Since $v\in\Omega^*(T^v)$ by $T^v=T(1+\mathscr{M})\supset 1+\mathscr{M}$ and $v_1\in\Omega^*(T_1)$ by Theorem 3.5, we have ρ is T^v-isotropic iff at least one residue class form $\bar{\rho}_a$ is $\bar{T}$-isotropic iff at least one residue class form $\bar{\rho}_a$ is $\bar{T}_1$-isotropic iff ρ is T_1-isotropic by Theorem 4.2, iff ρ is weakly isotropic by $T_1=\Sigma F_1^{2n}$.

(2) By Proposition 3.4 and since (F_1, v_1) is Henselian, $1+\mathscr{M}\subseteq F_1^{2n}\subseteq T_1$. So clearly $T\cdot(1+\mathscr{M})=T^v\subseteq F\cap T_1$. For the reverse inclusion, let $a\in\dot{F}\cap T_1$. Then $\langle 1, -a\rangle$ is isotropic so $\langle 1, -a\rangle$ is weakly isotropic. By (1), $\langle 1, -a\rangle$ is T^v-isotropic, hence we have $a\in T^v$.

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