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CYCLIC MENDELSOHN TRIPLE SYSTEMS*

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1. Introduction

A *cyclic triple* of a set S is a set of three ordered pairs of the form $\{(x,y), (y,z), (z,x)\}$ where x, y and z are distinct elements of S, and we will always denote the cyclic triple $\{(x,y), (y,z), (z,x)\}$ by (x,y,z), (y,z,x) or (z,x,y). A *Mendelsohn triple system* $MTS_\lambda(v)$ is a pair (V,B) where V is a v-set and B is a collection of cyclic triples of V (called blocks) such that every ordered pair of any two distinct elements of V is contained in exactly λ blocks of B. At first, Mendelsohn [6] introduced such systems with $\lambda=1$. However, in the case $\lambda>1$, we call those Mendelsohn triple systems as well. It is well-known [6] that a $MTS_\lambda(v)$ with $\lambda=1$ exists if and only if $v\equiv 1, 0 \pmod 3$ and $v\neq 6$. The spectrum of $MTS_\lambda(v)$ for every λ is $\lambda v(v-1)\equiv 0 \pmod 3$, except for the nonexisting system $MTS_1(6)$, which is determined by Bennett [1]. A $MTS_\lambda(v)$ is said to be *cyclic* if it admits an automorphism α consisting of a single cycle of the length v; and α is called a *cyclic automorphism.* Colbourn and Colbourn [2] have shown that a cyclic $MTS_\lambda(v)$ with $\lambda=1$ exists if and only if $v\equiv 1, 3 \pmod 6$ and $v\neq 9$.

In this paper, we obtain a necessary condition for the existence of a cyclic $MTS_\lambda(v)$ for every λ and show that this necessary condition is also sufficient, except possibly for either $\lambda=2$ and $v\equiv 6 \pmod{12}$, $v\neq 6$, or $\lambda=6$ and $v\equiv 2 \pmod{12}$.

A *triple system* $TS_\lambda(v)$ is a pair (V,B) where V is a v-set and B is a collection of 3-subsets of V (called blocks) such that every 2-subset of V is contained in precisely λ blocks of B. It is said that a $TS_\lambda(v)$ is a balanced incomplete block design with v elements, block size 3, and balance factor λ. Hanani [4] completely determined the existence of $TS_\lambda(v)$ for every λ. Cyclic $TS_\lambda(v)$ for every λ were settled by Colbourn

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and Colbourn [3].

THEOREM 1[3]. *There exists a cyclic* $TS_\lambda(v)$ *if and only if*
(i) $\lambda\equiv 1, 5, 7, 11 \pmod{12}$ *and* $v\equiv 1, 3 \pmod 6$ *or*
(ii) $\lambda\equiv 2, 10 \pmod{12}$ *and* $v\equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ *or*
(iii) $\lambda\equiv 3, 9 \pmod{12}$ *and* $v\equiv 1 \pmod 2$ *or*
(iv) $\lambda\equiv 4, 8 \pmod{12}$ *and* $v\equiv 0, 1 \pmod 3$ *or*
(v) $\lambda\equiv 6 \pmod{12}$ *and* $v\equiv 0, 1, 3 \pmod 4$ *or*
(vi) $\lambda\equiv 0 \pmod{12}$ *and* $v\geq 3$.
except for the nonexisting cyclic systems $TS_1(9)$ *and* $TS_2(9)$.

Throughout the paper, we will assume the set of elements of our cyclic $MTS_\lambda(v)$ to be the additive group $Z_v=\{0, 1, \cdots, v-1\}$ of residue classes of all integers Z modulo v and the corresponding cyclic automorphism to be the cycle $\alpha=(0, 1, \cdots, v-1)$ of the length v. We denote $Z_v^*=\{1, 2, \cdots, v-1\}$. For a positive integer n, denote nZ_v^* the multiset obtained from Z_v^* replicating each element n times.

2. The Necessary Condition for the Existence of Cyclic $MTS_\lambda(v)$

First of all, an elementary necessary condition for the existence of a cyclic $MTS_\lambda(v)$ is
(i) $\lambda\equiv 1, 2 \pmod 3$ and $v\equiv 0, 1 \pmod 3$ or
(ii) $\lambda\equiv 0 \pmod 3$ and $v\geq 3$,
except for $v=6$ with $\lambda=1$, since this is the spectrum for $MTS_\lambda(v)$.

In a cyclic $MTS_\lambda(v)$, if we regard each block (x, y, z) with a 3-set $\{x, y, z\}$ then the cyclic $MTS_\lambda(v)$ becomes a cyclic $TS_{2\lambda}(v)$. Thus, the following lemma immediately follows from Theorem 1.

LEMMA 2. *If either* $\lambda\equiv 1, 5 \pmod 6$ *and* $v\equiv 6, 10 \pmod{12}$, *or* $\lambda\equiv 3 \pmod 6$ *and* $v\equiv 2 \pmod 4$, *then there exists no cyclic* $MTS_\lambda(v)$.

Before proceeding to a stronger necessary condition, we need some observations concerning the structure of a cyclic $MTS_\lambda(v)$. Now restrict attention to cyclic $MTS_\lambda(v)$. For a block $b=(x, y, z)$, the set $C(b)=\{(x+i,\ y+i,\ z+i)\,|\,i=0, 1, \cdots, v-1;$ addition performed modulo $v\}$ is called the *orbit* of b and as element of $C(b)$ is called a *starter* block. Obviously, a collection of starter blocks taken one element from each orbit represents a cyclic $MTS_\lambda(v)$. Each block b has $|C(b)|\frac{v}{3}$ or v. In the former case, the block b is contained in either $C\left(\left(0, \frac{v}{3}, \frac{2v}{3}\right)\right)$ or $C\left(\left(0, \frac{2v}{3}, \frac{v}{3}\right)\right)$. Furthermore, each starter block (x, y, z) is uniquely associated with the 3-tuple $[a, b, c]$ (called a difference triple) where $a\equiv y-x \pmod{v}$, $b\equiv z-y \pmod{v}$ and $c\equiv x-z \pmod{v}$. Let $[a, b, c]$ be a difference triple obtained in this manner. It is evident that $a+b+c \equiv 0 \pmod{v}$. Consider a collection of difference triples which represent a cyclic $\mathrm{MTS}_\lambda(v)$; the collection of all differences in the difference triples is the multiset λZ_v^* if $v\equiv 1, 2 \pmod{3}$, or is the multiset $\lambda\left(Z_v^*-\left\{\frac{v}{3}, \frac{2v}{3}\right\}\right)\cup n\left\{\frac{v}{3}\right\}\cup m\left\{\frac{2v}{3}\right\}$ if $v\equiv 0 \pmod{3}$ where n and m are some positive integers such that $n+m\equiv 0 \pmod{3}$. These simple observations enable us to prove the following lemmas.

LEMMA 3. *If either* $\lambda\equiv 1, 5 \pmod{6}$ *and* $v\equiv 4 \pmod{12}$, *or* $\lambda=3 \pmod{6}$ *and* $v\equiv 4, 8 \pmod{12}$, *then there exists no cyclic* $\mathrm{MTS}_\lambda(v)$.

Proof. Suppose there exists a cyclic $\mathrm{MTS}_\lambda(v)$ for either $\lambda\equiv 1, 5 \pmod{6}$ and $v\equiv 4 \pmod{12}$, or $\lambda\equiv 3 \pmod{6}$ and $v\equiv 4, 8 \pmod{12}$, and consider the collection of difference triples which represent the cyclic $\mathrm{MTS}_\lambda(v)$. Then the multiset λZ_v^* is the collection of all differences in the difference triples. Since v divides the sum of the three differences in each difference triple, v divides the sum of all integers in λZ_v^* and hence $\frac{1}{2}\lambda v(v-1)\equiv 0 \pmod{v}$, which is impossible since λ and $v-1$ are odd integers.

LEMMA 4. *If either* $\lambda\equiv 1, 5 \pmod{6}$ *or* $\lambda\equiv 3 \pmod{6}$ *and* $v\equiv 0 \pmod{12}$, *then there exists no cyclic* $\mathrm{MTS}_\lambda(v)$.

Proof. As proof of Lemma 3, in this case, v should divide the sum of all integers in the multiset

$$\lambda\left(Z_v^*-\left\{\frac{v}{3}, \frac{2v}{3}\right\}\right)\cup n\left\{\frac{v}{3}\right\}\cup m\left\{\frac{2v}{3}\right\}$$

where n and m are some positive integers such that $n+m\equiv 0 \pmod{3}$. Thus, $\frac{1}{2}\lambda v(v-1)+(n-\lambda)\left\{\frac{v}{3}\right\}+(m-\lambda)\left\{\frac{2v}{3}\right\}\equiv 0 \pmod{v}$, which is impossible since both λ and $v-1$ are odd integers.

Since there exists no $MTS_1(6)$, so does no cyclic $MTS_1(6)$, and it is straightforward that there exists no cyclic $MTS_1(9)$ [also, see 2, 3, 5]. Moreover, a simple argument demonstrates that there exist neither cyclic $MTS_2(6)$ nor cyclic $MTS_2(9)$. Summarizing, we have the following necessary condition.

THEOREM 5. *If there exists a cyclic* $MTS_\lambda(v)$, *then*
(i) $\lambda\equiv 1, 5 \pmod 6$ *and* $v\equiv 1, 3 \pmod 6$ *or*
(ii) $\lambda\equiv 2, 4 \pmod 6$ *and* $v\equiv 0, 1 \pmod 3$ *or*
(iii) $\lambda\equiv 3 \pmod 6$ *and* $v\equiv 1 \pmod 2$ *or*
(iv) $\lambda\equiv 0 \pmod 6$ *and* $v\geq 3$,
except for each pair $\{\lambda, v\}$ *of* $\lambda=1$ *or* 2 *and* $v=6$ *or* 9.

3. The Existence of Cyclic $MTS_\lambda(v)$

Since the existence of a cyclic $TS_\lambda(v)$ implies the existence of a cyclic $MTS_\lambda(v)$ by replacing each block $\{x, y, z\}$ with the two cyclic triples (x, y, z) and (x, z, y). Thus, we have the following lemma from Theorem 1.

LEMMA 6. *There exists a cyclic* $MTS_\lambda(v)$ *for the following values* λ *and* v:
(i) $\lambda\equiv 1, 5, 7, 11 \pmod{12}$ *and* $v\equiv 1, 3 \pmod 6$ *or*
(ii) $\lambda\equiv 2, 10 \pmod{12}$ *and* $v\equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ *or*
(iii) $\lambda\equiv 3, 9 \pmod{12}$ *and* $v\equiv 1 \pmod 2$ *or*
(iv) $\lambda\equiv 4, 8 \pmod{12}$ *and* $v\equiv 0, 1 \pmod 3$ *or*
(v) $\lambda\equiv 6 \pmod{12}$ *and* $v\equiv 0, 1, 3 \pmod 4$ *or*
(vi) $\lambda\equiv 0 \pmod{12}$ *and* $v\geq 3$,
except for the nonexisting cyclic systems $MTS_1(9)$ *and* $MTS_2(9)$.

So as to be that the necessary condition in Theorem 5 is sufficient, it remains to construct a cyclic $MTS_\lambda(v)$ for the following values λ and v:
(i) $\lambda\equiv 2 \pmod{12}$ and $v\equiv 6, 10 \pmod{12}$, $v\neq 6$ or
(ii) $\lambda\equiv 0 \pmod 6$ and $v\equiv 2 \pmod 4$.
It is easy to see that if we replicate each block of a cyclic $MTS_\lambda(v)$ n times, then we will get a cyclic $MTS_{n\lambda}(v)$ for every positive integer n. For $v\equiv 6 \pmod{12}$, we have not succeeded to construct a cyclic $MTS_2(v)$ yet, but we construct a cyclic $MTS_6(v)$ in the following lemma.

LEMMA 7. *If* $v\equiv 6 \pmod{12}$, *then there exists a cyclic* $MTS_6(v)$.

Proof. Let $v\equiv 6 \pmod{12}$ and let

$$A=\left\{(0,r,v-r),\quad (0,r,v-r),\quad (0,v-r,r)\,|\,r=1,2,\cdots,\frac{v}{3}-1\right\},$$

$$B=\left\{\left(0,\ \frac{2v}{3}+1-r,\ r\right),\ \left(0,\ \frac{2v}{3}+1-r,\ r\right),\right.$$
$$\left.\left(0,\ r,\ \frac{2v}{3}+1-r\right)\Big|\,r=1,\ 2,\ \cdots,\ \frac{v}{3}\right\},$$

$$C=\left\{\left(0,\ \frac{v}{3},\ \frac{2v}{3}\right),\ \left(0,\ \frac{v}{3},\ \frac{2v}{3}\right),\ \left(0,\ \frac{v}{3},\ \frac{2v}{3}\right)\right\}.$$

Then $A\cup B\cup C$ is a collection of starter blocks which represent a cyclic $MTS_6(v)$.

LEMMA 8. *If* $v\equiv 10 \pmod{12}$, *then there exists a cyclic* $MTS_2(v)$.

Proof. If $v\equiv 10 \pmod{12}$, then

$$\left\{(0,\ r,\ v-r),\ \left(0,\ \frac{2(v-1)}{3}+1-r,\ r\right)\Big|\,r=1,2,\cdots,\ \frac{v-1}{3}\right\}$$

is a collection of starter blocks which represent a cyclic $MTS_2(v)$.

The remaining values for which the existence of a cyclic $MTS_\lambda(v)$ is in doubt are

(i) $\lambda=2$ and $v\equiv 6 \pmod{12}$, $v\neq 6$ or

(ii) $\lambda=6$ and $v\equiv 2 \pmod{12}$.

However, the following examples suggest that for these values there seems to exist a cyclic $MTS_\lambda(v)$.

EXAMPLE 9. A cyclic $MTS_2(18)$ has starter blocks

(0, 16, 17), (0, 13, 16), (0, 5, 15), (0, 14, 3), (0, 9, 13),
(0, 11, 4), (0, 2, 10), (0, 17, 9), (0, 15, 1), (0, 7, 2),
(0, 6, 12), (0, 6, 12), (0, 12, 6), (0, 12, 6).

EXAMPLE 10. A cyclic $MTS_2(30)$ has starter blocks

(0, 17, 18), (0, 7, 28), (0, 22, 27), (0, 24, 26), (0, 25, 21),
(0, 23, 24), (0, 4, 23), (0, 24, 16), (0, 12, 3), (0, 11, 22),
(0, 25, 13), (0, 14, 27), (0, 29, 15), (0, 26, 1), (0, 19, 2),
(0, 23, 21), (0, 15, 3), (0, 16, 24), (0, 10, 20), (0, 10, 20),

(0, 20, 10), (0, 20, 10).

EXAMPLE 11. A cyclic $MTS_6(14)$ has starter blocks

(0, 3, 13), (0, 10, 12), (0, 8, 11), (0, 5, 10), (0, 1, 9),
(0, 13, 3), (0, 12, 10), (0, 11, 8), (0, 9, 1), (0, 7, 8),
(0, 8, 7), (0, 12, 13), (0, 3, 5), (0, 6, 9), (0, 7, 11),
(0, 13, 8), (0, 4, 10), (0, 10, 4), (0, 12, 7), (0, 12, 7),
(0, 2, 4), (0, 8, 13), (0, 2, 1), (0, 5, 3), (0, 9, 6),
(0, 11, 7).

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