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PROBABILISTIC INTERPRETATION OF NON-SYMMETRIC DIRICHLET FORMS

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1. Introduction

Analysis based on additive functionals (AF's for short) and stochastic calculus related to symmetric Dirichlet spaces were developed by M. Fukushima [3], S. Nakao [7], M. Takeda [9] and others. Many results in the symmetric case have been extended to the non-symmetric case by S. Carrillo Menendez [2], J.H. Kim [4], Y. Le Jan [5] etc.. The purpose of this paper is to give a functional calculus on Dirichlet spaces (Theorem 3.4) and a general form of below (2.6).

We first give the definition of non-symmetric Dirichlet space. Let X be a locally compact Hausdorff space with countable base and m a non-negative Radon measure on X such that supp $[m]=X$. $L^2(X, m)$ denotes the real L^2-space with inner product

$$(u, v)_{L^2}=\int_X u(x)v(x)m(dx), \quad u, v\in L^2(X, m).$$

Let H be a dense linear subspace of $L^2(X, m)$ which forms a Hilbert space with a norm $\| \ \|_H$ such that for some $K>0$, $\|u\|_H \geqq K\|u\|_{L^2}$ for any $u\in H$. Moreover we assume that if $u\in H$, then $|u|$, $u\wedge 1\in H$. In this article we consider a bilinear form a on $H\times H$ which satisfies the following conditions;

(a. 1) there exists a constant $\alpha_0\geqq 0$ such that a_α is coercive for any $\alpha>\alpha_0$, i. e., for some constant $K_1=K_1(\alpha)>0$, $a_\alpha(u, u)\geqq K_1\|u\|_H^2$ for every $u\in H$,

(a. 2) a is continuous in the following sense; there exists a constant $K_2>0$ such that $|a(u, v)|\leqq K_2\|u\|_H\|v\|_H$ for all $u, v\in H$,

(a. 3) $a(T_1u, u-T_1u)\geqq 0$,

(a. 4) $\hat{a}(T_1u, u-T_1u)\geqq 0$.

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Here $a_\alpha(u, v)=a(u, v)+\alpha(u, v)_{L^2}$, $\hat{a}(u, v)=a(v, u)$ and $T_1u=u^+\wedge 1$ ($u^+=u\vee 0$). A bilinear form a which fulfills (a. 1)~(a. 4) is called a *Dirichlet form* on $H\times H$, and (a, H) a *Dirichlet space* on $L^2(X, m)$. Let $C_0(X)$ be the set of all bounded continuous functions on X with compact support. From now on we assume that H is *reqular*, i. e., $C_0(X)\cap H$ is dense in H with the norm $\| \|_H$ and dense in $C_0(X)$ with the uniform norm.

J. H. Kim [4] gave the Beurling–Deny formula for a; for every $u, v\in H$,

(1. 1) $\frac{1}{2}[a(u, v)+\hat{a}(u, v)]=a^c(u, v)+a^j(u, v)+a^k(u, v)+a^{\hat{k}}(u, v)$,

where $a^c(u, v)$ is a symmetric form on $H\times H$ such that for $u, v\in H$ with compact support, $a^c(u, v)=0$ if v is constant on a neighbourhood of supp $[u]$ (=support of u), we say that the form a^c (,) satisfies the *local property*, and

(1. 2) $a^j(u, v)=\frac{1}{2}\int_{X\times X-\varDelta}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))\sigma(dx, dy)$,

(1. 3) $a^k(u, v)=\frac{1}{2}\int_X \tilde{u}(x)\tilde{v}(x)\chi(dx)$, $a^{\hat{k}}(u, v)=\frac{1}{2}\int_X \tilde{u}(x)\tilde{v}(x)\hat{\chi}(dx)$.

Here σ is a positive Radon measure on $X\times X-\varDelta$ ($\varDelta$ is diagonal) satisfying

$$\int_{X\times X-\varDelta}u(x)v(y)\sigma(dx, dy)=-a(u, v)$$

for $u, v\in C_0(X)\cap H$ such that $\mathrm{supp}[u]\cap\mathrm{supp}[v]=\phi$, and χ and $\hat{\chi}$ are positive Radon measures on X satisfying

$$\int_X u(x)\chi(dx)=a(v, u)-\int_{X\times X-\varDelta}u(y)(1-v(x))\sigma(dx, dy)$$

and

$$\int_X u(x)\hat{\chi}(dx)=a(v, u)-\int_{X\times X-\varDelta}u(x)(1-v(y))\sigma(dx, dy)$$

for $u, v\in C_0(X)\cap H$ such that $v=1$ on a neighbourhood of $\mathrm{supp}[u]$, and $\tilde{u}$ is a q. c. version of $u\in H$ (see [3] for the definition).

2. Probabilistic interpretation of (1. 1)

In this section we give a probabilistic interpretation of each term of the right hand side of (1. 1) and an example.

Let $M=(P_x, X_t)$ and $\hat{M}=(\hat{P}_x, \hat{X}_t)$ be the *Hunt processes* associated with a and $\hat{a}$, respectively, $\boldsymbol{S}$ the set of all *smooth measures* of $\boldsymbol{M}$ and $\boldsymbol{A}_c^+$ the set of all *positive continuous* AF's of $\boldsymbol{M}$ (see [3] for the definitions). S. Carrillo Menendez [1] showed that $\boldsymbol{S}$ and $\boldsymbol{A}_c^+$ are in one to one correspondence which is characterized by the following relation; for $A\in\boldsymbol{A}_c^+$ and $\mu\in\boldsymbol{S}$,

$$\int_X E_x[\int_0^t f(X_s)dA_s]h(x)m(dx)=\int_0^t\{\int_X \hat{E}_x[h(\hat{X}_s)]f(x)\mu(dx)\}\,ds$$

for any non-negative Borel functions f and h on X and $t>0$, where $E_x[Y]$ (resp. $\hat{E}_x[Y]$) is the expectation of random variable Y with respect to P_x (resp. $\hat{P}_x$). J. H. Kim [4] defined the *energy* of AF A by

$$e(A)=\lim_{\alpha\to\infty}\frac{\alpha^2}{2}\int_X E_x[\int_0^\infty e^{-\alpha t}A_t{}^2dt]m(dx) \tag{2.1}$$

if the limit exists, and defined the *mutual energy* of AF's A and B by

$$e(A, B)=\frac{1}{2}(e(A+B)-e(A)-e(B))$$

and proved that the set $\mathring{\mathcal{M}}$ of all martingale AF's of finite energy is a real Hilbert space with $e(,)$ and that the AF

$$A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)\quad(u\in H) \tag{2.2}$$

has a unique decomposition

$$A^{[u]}=M^{[u]}+N^{[u]},\quad M^{[u]}\in\mathring{\mathcal{M}},\ N^{[u]}\in\mathcal{N}_c, \tag{2.3}$$

where $\mathcal{N}_c$ is the set of all continuous AF's of zero energy. M. Fukushima [3] proved the above results in the symmetric case. It is known that $M^{[u]}$ has a decomposition

$$M^{[u]}=M^{c[u]}+M^{d[u]}=M^{c[u]}+M^{j[u]}+M^{k[u]} \tag{2.4}$$

where $M^{c[u]}$ and $M^{d[u]}$ are the continuous and purely discontinuous part of $M^{[u]}$ respectively, and $M^{j[u]}$ and $M^{k[u]}$ are defined by

$$M^{k[u]}=-\widehat{\tilde{u}(X_{\zeta-})I_{\{\zeta\leq t\}}},\quad M^{j[u]}=M^{d[u]}-M^{k[u]}.$$

Here, ζ is the life time of $\boldsymbol{M}$ and for AF A, $\widehat{A}$ denotes $A-A^p$ with A^p being the dual predictable projection of A (cf. [2]). The smooth measure corresponding to $A\in\boldsymbol{A}_c^+$ is denoted by μ_A. If $A_t=A_t{}^{(1)}-A_t{}^{(2)}$ a. s., $A^{(i)}\in\boldsymbol{A}_c^+$ and $\mu_{A^{(i)}}$, $i=1, 2$, are bounded measures, then the measure μ_A corresponding to A is given by $\mu_{A^{(1)}}-\mu_{A^{(2)}}$. In particular, for $M^{[u]}, M^{[v]}\in\mathring{\mathcal{M}}(u, v\in H)$, we use the abbreviations $\mu_{\langle u,v\rangle}$ and $\mu^i{}_{\langle u,v\rangle}$ for $\mu_{\langle M^{[u]},M^{[v]}\rangle}$ and $\mu_{\langle M^{i[u]},M^{i[v]}\rangle}$, $i=c, d, j, k$, respectively. The symbol $\langle M, N\rangle$ is the quadratic variation of the martingales M and N. Then we have the following result.

THEOREM 2.1 ([4]). (i) *For any* $u, v\in H$, *we have*

$$\frac{1}{2}\mu^i_{<u,v>}(X)=a^i(u,v) \quad for\ i=c, j, \hat{k}, k. \tag{2.5}$$

(ii) *For* $u, v\in H_b$(*=the set of all bounded functions in* H) *and* $w\in H$,

$$d\mu^c_{<uv,w>}(x)=\tilde{u}(x)d\mu^c_{<v,w>}(x)+\tilde{v}(x)d\mu^c_{<u,w>}(x)\ (x\in X). \tag{2.6}$$

The equality (2.6) is essential to prove a strong local property of the symmetric form a^c (,) (see Corollary 5.10 in [4]). In the next section we give a functional rule of $\mu^c_{<u,v>}$ as an extension of (2.6).

EXAMPLE 2.2. Let dx be the Lebesgue measure on d-dimensional Euclidean space R^d and $H^1(R^d)$ the Sobolev space of order 1, i.e.,

$$H^1(R^d)=\{u\in L^2(R^d, dx) : \frac{\partial u}{\partial x_i}\in L^2(R^d, dx), 1\leqq i\leqq d\},$$

where the derivatives $\frac{\partial u}{\partial x_i}$ are taken in the sence of Schwartz distributions. We define the norm on $H^1(R^d)$ by, for $u\in H^1(R^d)$,

$$\|u\|_{H^1}=\|u\|_{L^2}+\|u_x\|_{L^2},$$

where

$$\|u_x\|_{L^2}=\left(\sum_{i=1}^{d}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2}\right)^{1/2}.$$

Consider the following differential operator given formally by

$$\begin{aligned} L&=L^0+\sum_{i\neq j}^{d}\delta(x_i-x_j)\frac{\partial}{\partial x_i}\\ &=L^0+\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}H(x_j-x_i)\frac{\partial}{\partial x_i}, \end{aligned}$$

where $H(y)=\frac{1}{2}(I_{\{y>0\}}-I_{\{y<0\}})$ and

$$L^0u=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{d}b_i\frac{\partial u}{\partial x_i}-cu.$$

Suppose that a_{ij}, b_i, $i, j=1, 2, \ldots, d$, and c are bounded measurable functions on R^d satisfying that

(i) there exists a constant $\nu>0$ such that

$$\sum_{i,j=1}^{d}a_{ij}(x)y_iy_j\geqq\nu|y|^2 \text{ for all } y\in R^d,$$

(ii) $c \geqq 0$,

(iii) $c+\sum_{i=1}^{d}(b_i)_{x_i} \geqq 0$ (in the sense of distributions).

Then the bilinear form a corresponding to L is given by, for $u, v \in H^1(R^d)$,

$$(2.7) \quad a(u,v)=\sum_{i,j=1}^{d}\int_{R^d}\overline{a_{ij}}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx-\sum_{i=1}^{d}\int_{R^d}b_i v\frac{\partial u}{\partial x_i}dx+\int_{R^d}cuv\,dx,$$

where $\overline{a_{ij}}(x)=a_{ij}(x)+H(x_j-x_i)$, $i,j=1,2,\ldots,d$. And $(a, H^1(R^d))$ is a regular Dirichlet space (see [4]). In this case

$$\mu^c_{<u,v>}(dx)=\sum_{i,j=1}^{d}(\overline{a_{ij}}+\overline{a_{ji}})\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx, \quad \mu^j_{<u,v>}(dx)=0,$$

$$\mu^k_{<u,v>}(dx)=cuvdx\ , \quad \mu^k_{<u,v>}(dx)=uv(c+\sum_{i=1}^{d}(b_i)_{x_i})dx.$$

REMARK. The process corresponding to the form a defined by (2.7) is special case of the diffusion treated by H. Osada [8]. He has given a specific construction of the diffusion by means of the associated transition density function.

3. Functional rule of $\mu^c_{<\cdot,\cdot>}$

Let $(G_\alpha)_{\alpha>\alpha_0}$ (resp. $(\hat{G}_\alpha)_{\alpha<\alpha_0}$) is the *resolvent* of a (resp. $\hat{a}$) (see [3] or [5]).

LEMMA 3.1 ([3]). *If the resolvent* $(G_\alpha)_{\alpha>\alpha_0}$ *is positive, then there exists a unique measure* σ_α *on* $X\times X$ *such that*

$$\alpha^2(G_\alpha u, v)_{L^2}=\int_{X\times X}u(x)v(x)\sigma_\alpha(dx,dy)$$

for every Borel functions $u, v\in L^2(X,m)$.

Let

$$a^{(\alpha)}(u,v)=\alpha(u-\alpha G_\alpha u, v)_{L^2}$$

and

$$\hat{a}^{(\alpha)}(u,v)=\alpha(u-\alpha\hat{G}_\alpha u, v)_{L^2}$$

for any $\alpha>\alpha_0$ and $u, v\in L^2(X,m)$.

LEMMA 3.2 ([5]). (i) *Let* $u\in L^2(X,m)$. *Then* $u\in H$ *if and only if* $\sup_{\alpha>\alpha_0} a^{(\alpha)}(u,v)<\infty$,

(ii) *For any* $u, v\in H$, $\lim_{\alpha\to\infty} a^{(\alpha)}(u,v)=a(u,v)$.

LEMMA 3. 3. *For any Borel fonctions* $u, v \in L^2(X, m)$,

$$a^{(\alpha)}(u, v) + \hat{a}^{(\alpha)}(u, v) = \int_{X\times X} (u(x)-u(y))(v(x)-v(y))\sigma_\alpha(dx, dy) + \int_X u(x)v(x)(1-\alpha G_\alpha 1)(x)m(dx) + \int_X u(x)v(x)(1-\alpha \hat{G}_\alpha 1)(x)m(dx). \tag{3.1}$$

Proof. By Lemma 3. 1 and the definitions of $a^{(\alpha)}$ and $\hat{a}^{(\alpha)}$, we can prove that for any $u, v \in L^2(X, m)$

$$a^{(\alpha)}(u, v) = \int_{X\times X} u(x)(v(x)-v(y))\sigma_\alpha(dx, dy) + \int_X u(x)v(x)(1-\alpha \hat{G}_\alpha 1)(x)m(dx)$$

and

$$\hat{a}^{(\alpha)}(u, v) = \int_{X\times X} u(x)(v(x)-v(y))\sigma_\alpha(dx, dy) + \int_X u(x)v(x)(1-\alpha G_\alpha 1)(x)m(dx)$$

Form these we easily see that (3. 1) holds. The proof is complete.

Now we give a functional rule on Dirichlet spaces.

THEOREM 3. 4. *Let* $\Phi \in C^1(R^m)$ *and* $\Phi(0)=0$. *For any* $u_1, u_2, \ldots, u_m \in H_b$, *the composition function* $\Phi(\mathrm{u}) = \Phi(u_1, u_2, \ldots, u_m)$ *belong to* H_b *and*

$$a_\alpha(\Phi(\mathrm{u}), \Phi(\mathrm{u})) \leqq \frac{m+2}{2} \sum_{i=2}^{m} \|\Phi_{x_i}\|^2_{L^\infty(V)} a_\alpha(u_i, u_i), \tag{3.2}$$

where V *is an* m*-dimensional finite cube containing the range of* $\mathrm{u}(x) = (u_1(x), u_2(x), \ldots, u_m(x))$ $(x \in X)$.

Proof. Let $u^{(1)}, u^{(2)}, \ldots, u^{(m)} \in H$ and $w \in L^2(X, m)$ satisfy

$$|w(x)| \leqq \sum_{i=1}^{m} |u^{(i)}(x)|, \quad |w(x)-w(y)| \leqq \sum_{i=1}^{m} |u^{(i)}(x)-u^{(i)}(y)|. \tag{3.3}$$

Using w in place of u and v in (3. 1), we find that

$$a^{(\alpha)}(w, w) \leqq \frac{m+2}{2} \sum_{i=1}^{m} a^{(\alpha)}(u^{(i)}, u^{(i)}). \tag{3.4}$$

Applying the mean value theorem, we can prove that $w = \Phi(\mathrm{u})$ and $u^{(i)} = \|\Phi_{x_i}\|_{L^\infty(V)} u_i$, $i = 1, 2, \ldots, m$, satisfy (3. 3). And hence

$$a^{(\alpha)}(\Phi(\mathrm{u}), \Phi(\mathrm{u})) \leq \frac{m+2}{2} \sum_{i=1}^{m} \|\Phi_{x_i}\|^2_{L^\infty(V)} a^{(\alpha)}(u_i, u_i).$$

This inequality and Lemma 3.2 complete the proof.

COROLLARY 3.5. *For any $v \in H_b$, we have*

$$d\mu^c_{<\Phi(\mathrm{u}), v>} = \sum_{i=1}^{m} \Phi_{x_i}(\mathrm{u})\, d\mu^c_{<u_i, v>}.$$

Proof. This follows from Theorem 3.4 by the same method as Theorem 5.4.1 in [3].

REMARK. Let $m=1$ in Corollary 3.5. Then the image measure $\mu^c_{<u,u>}(u^{-1}(\cdot))$ of $\mu^c_{<u,u>}$ induced by $u \in H$ is absolutely continuous with respect to Lebesgue measure ν on R^1, i.e., if $\nu(E)=0$, then $\mu^c_{<u,u>}(u^{-1}(E))=0$. In fact, let $\Phi(t)=\int_0^t I_E(s)\,d\nu(s)$ and assume that $\nu(E)=0$. Then

$$\begin{aligned}\mu^c_{<u,u>}(u^{-1}(E)) &= \int_X I_{u^{-1}(E)}(x)\, d\mu^c_{<u,u>}(x) \\ &= \int_X I_E(u(x))\, d\mu^c_{<u,u>}(x) \\ &= \int_X \Phi'(u(x))\ \ d\mu^c_{<u,u>}(x)\end{aligned}$$

By Corollary 3.5, the last term is equal to $\mu^c_{<\Phi(u), u>}(X)$ and hence equal to 0 since $\Phi(u(x))=0$ for all $x \in X$.

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