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HOMOLOGY AND GENERALIZED EVALUATION SUBGROUPS OF HOMOTOPY GROUPS

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D. H. Gottlieb showed that $G_n(X)$ is contained in the kernel of mod p Hurewicz homomorphism under some conditions. He proved the following theorems [2].

THEOREM. *Let X be a topological space with finitely generated integer homology. If n is an odd integer, then $G_n(X)$ is contained in the kernel of h_p for p any prime number provided the Euler-Poincare number $\chi(X) \neq 0$, where $h_p : \Pi_n(X) \to H_n(X ; Z_p)$ is the map as composition of Hurewicz map tensored with Z_p.*

THEOREM. *Let X have finitely generated integer homology. Suppose p is a prime which does not divided $\chi(X)$. Then $G_n(X)$ is contained in the kernel of h_p for even n.*

In [5], the first author and Kim introduced the generalized evaluation subgroups $G_n^f(X, A)$ of homotopy groups $\Pi_n(X)$ as a generalization of $G_n(X)$.

Let $(X, *)$ and $(A, *)$ be any two pointed topological spaces and $f : (A, *) \to (X, *)$ be a fixed map.
Consider the class of continuous functions

$$\phi : A \times S^n \to X$$

such that $\phi(a, *) = f(a)$. Then the map $g : (S^n, *) \to (X, *)$ defined by $g(s) = \phi(*, s)$ represents an element $[g] \in \Pi_n(X, *)$. The set of all element $[g] \in \Pi_n(X, *)$ obtained in the above manner from some ϕ was denoted by $G_n^f(X, A, *)$ and called the *generalized evaluation subgroup* of $\Pi_n(X, *)$. Thus for every $[g] \in G_n^f(X, A, *)$, there is at least one map $\phi : A \times S^n \to X$ which satisfies the above conditions. We say that ϕ is an *affiliated map* to $[g]$ with respect to A. It was denoted by $G_n(X, A)$ if f is an inclusion from A to X. The evaluation subgroup

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$G_n(X)$ is equal to $G_n^1(X, X, *)$.

Since $G_n(X, A)$ contains $G_n(X)$, we raise some questions; Is $G_n(X, A)$ contained in the kernel of h_p? If it is not true, what are conditions so that it is true?

In this paper, in order to solve the problem, we study the algebraic structure induced by $\phi : A\times S^n \to X$ affiliated to some $\alpha\in G_n^f(X, A)$, on the homology which is a modified notion of D. H. Gottlieb's results. And we show that it is closely related to $G_n^{Rel}(X, A)$ constructed by us [6].

By the künneth formula and the fact that $H_*(S^n, Z)$ has no torsion, we have

$$\mu : H_*(A ; G)\otimes H_*(S_n ; Z) \cong H_*(A\times S^n ; G).$$

Thus if $x\in H_*(A\times S^n ; G)$, $x=\mu(y\otimes 1+z\otimes\lambda)$, for some $y, z\in H_*(A ; G)$ where $\lambda\in H_n(S^n ; Z)$ is a fundamental class of S^n. We shall denote $\mu(z\otimes z')$ by $z\times z'$.

PROPOSITION 1. *Let $\phi : A\times S^n\to X$ be an affiliated map of $\alpha\in G_n^f(X, A, x_0)$ with trace g. Then in homology, $\phi_*(1\times\lambda)=g_*(\lambda)$ and if f has a left homotopy inverse r, $r_*\phi_*(x\times 1)=x$, for $x\in H_n(A ; G)$.*

Proof. Let $i_1 : A\to A\times S^n$ be the map which is given by $i_1(x)=(x, s_0)$ and $i_2 : S^n\to A\times S^n$ given by $i_2(s)=(x_0, s)$, where x_0, s_0 are base points of A, S^n, respectively. Let p_1 and p_2 be the natural projections from $A\times S^n$ to A, to S^n, respectively. Then $p_{1*}(z\times z')=z\times\varepsilon(z')$ and $p_{2*}(z\times z')=\varepsilon(z)\times z'$, where ε is an augumentation. See [3]. Thus $p_{1*}(x\times 1)=x$ and $p_{1*}(x\times\lambda)=0$. Also $p_{2*}(1\times\lambda)=\lambda$, $p_{2*}(x\times\lambda)=0$ and $p_{2*}(x\times 1)=0$ if $x\in H_q(A ; G)$ where $q>0$. Now since $p_1 i_1=1_A$, $i_{1*}(x)=x\times 1$ for $x\in H_*(A ; G)$ and since $p_2 i_2=1_{S^n}$, $i_{2*}(y)=1\times y$, for $y\in H_*(S^n ; Z)$. Since $\phi i_2=g$, $\phi_*(1\times\lambda)=\phi_* i_{2*}(\lambda)=g_*(\lambda)$. Since $\phi i_1=f$, $\phi_*(x\times 1)=\phi_* i_{1*}(x)=f_*(x)$. Thus $r_*\phi_*(x\times 1)=r_*\phi_* i_{1*}(x)=r_*f_*(x)=x$.

PROPOSITION 2. *Let $\phi : A\times S^n\to X$ be an affiliated map of $\alpha\in G_n^f(X, A, x_0)$ with trace g. If r is a left homotopy inverse of f, then $r\phi$ induces a homomorphism*

$$K_\lambda : H_q(A ; G)\to H_{q+n}(A ; G).$$

Proof. Define $K_\lambda(x)=r_*\phi_*(x\times\lambda)$. Then K_λ is a homomorphism. In fact, $K_\lambda(x+y)=r_*\phi_*((x+y)\times\lambda)=r_*\phi_*(\mu((x+y)\otimes\lambda))$

$$=r_*\phi_*(\mu(x\otimes\lambda)+\mu(y\otimes\lambda))=r_*\phi_*(x\otimes\lambda)+r_*\phi_*(y\times\lambda)$$
$$=K_\lambda(x)+K_\lambda(y).$$

We shall define $K_\lambda{}^n(x)=K_\lambda(K_\lambda{}^{n-1}(x))$ and $K_\lambda{}^0(x)=r_*\phi_*(x\times 1)$. Then $K_\lambda{}^0(x)=x$ by Proposition 1.

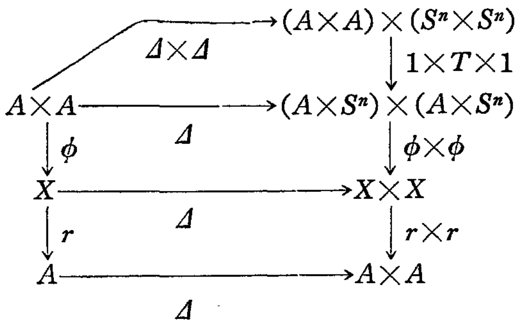
Let G be a field. Then by the künneth formula,

$$\mu' : H_*(A\,;G)\otimes H_*(A\,;G)=H_*(A\times A\,;G).$$

We shall denote $\mu'(z\otimes z')$ by $z\times z'$. If Δ stand for the diagonal map $A\to A\times A$, then $\Delta_*(x)=\sum z_i\times z_i'$ where $z_i, z_i'\in H_*(A\,;G)$.

PROPOSITION 3. *Let Δ be the diagonal map $A\to A\times A$ and $\Delta_*(x)=\sum z_i\times z_i'$. Then $\Delta_*(K_\lambda(x))=\sum z_i\times K_\lambda(z_i')+\sum(-1)^{n\dim z_i'}K_\lambda(z_i)\times z_i'$.*

Proof. Consider the following diagram;



The diagram commutes where $T: A\times S^n\to S^n\times A$ is given by $T(x,s)=(s,x)$, ϕ is an affiliated map and r is a left homotopy inverse of f. Note that $T_*: H_*(A\,;G)\otimes H_*(S^n\,;Z)\to H_*(S^n\,;Z)\otimes H_*(A\,;G)$ is given by $T_*(z\otimes x)=(-1)^{pq}x\otimes z$ where $z\in H_q(A\,;G)$ and $x\in H_q(S^n\,;Z)$. Thus

$$\Delta_*(K_\lambda(x))=\Delta_*r_*\phi_*(x\times\lambda)$$
$$=(r_*\times r_*)(\phi_*\times\phi_*)(1\times T_*\times 1)(\Delta_*\times\Delta_*)(x\times\lambda)$$
$$=(r_*\times r_*)(\phi_*\times\phi_*)(1\times T_*\times 1)(\Delta_*(x)\times(1\times\lambda+\lambda\times 1))$$
$$=(r_*\times r_*)(\phi_*\times\phi_*)(1\times T_*\times 1)(\sum(z_i\times z_i')\times(1\times\lambda+\lambda\times 1))$$
$$=(r_*\times r_*)(\phi_*\times\phi_*)(1\times T_*\times 1)((\sum(z_i\times z_i'\times 1\times\lambda)+(\sum(z_i\times z_i'\times\lambda\times 1))$$
$$=(r_*\times r_*)(\phi_*\times\phi_*)\{\sum(z_i\times 1)\times(z_i'\times\lambda)+\sum(-1)^{n\dim z_i'}(z_i\times\lambda)\times(z_i'\times 1)\}$$
$$=\sum r_*\phi_*(z_i\times 1)\times r_*\phi_*(z_i'\times\lambda)+\sum(-1)^{n\dim z_i'}r_*\phi_*(z_i\times\lambda)\times r_*\phi_*(z_i'\times 1)$$

$=\sum z_i\times K_\lambda(z_i')+\sum(-1)^{n\dim z_i'}K_\lambda(z_i)\times z_i'$.

We define the homomorphism $K_\lambda{}^p\otimes K_\lambda{}^q$, $p, q=0, 1, 2, \cdots$, on $H_*(A\,;G)\otimes H_*(A\,;G)$ by the rule

$(K_\lambda{}^p\otimes K_\lambda{}^q)(x\otimes y)=(-1)^{pn\dim y}K_\lambda{}^p(x)\otimes K_\lambda{}^q(y)$. We shall denote $\mu(K^p\otimes K_\lambda{}^q)=K_\lambda{}^p\times K_\lambda{}^q$. Thus, by Propostion 3, we see that

$$\varDelta_*(K_\lambda(x))=(K_\lambda\times 1+1\times K_\lambda)(\varDelta_*(x)).$$

Note that

$$\varDelta_*(K_\lambda{}^2(x))=\varDelta_*(K_\lambda(K_\lambda(x)))=(K_\lambda\times 1+1\times K_\lambda)(\varDelta_*(K_\lambda(x)))$$
$$=(K_\lambda\times 1+1\times K_\lambda)(K_\lambda\times 1+1\times K_\lambda)(\varDelta_*(x)).$$

So $\varDelta_*(K_\lambda{}^p(x))=(K_\lambda\times 1+1\times K_\lambda)^{p-1}(K_\lambda\times 1+1\times K_\lambda)\ \ (\varDelta_*(x))$.

Remark 1. If n is even, $(K_\lambda{}^p\times K_\lambda{}^q)(x\times y)=K_\lambda{}^p(x)\times K_\lambda{}^q(y)$. So we may regard $(K_\lambda\times 1+1\times K_\lambda)^p=\sum\binom{p}{i}K_\lambda{}^i\times K_\lambda{}^{p-i}$.

On the other hand, if n is odd, then observe that

$$\begin{aligned}(K_\lambda\times 1+1\times K_\lambda)^2(x\times y)&=(K_\lambda\times 1+1\times K_\lambda)(K_\lambda\times 1+1\times K_\lambda)(x\times y)\\&=(K_\lambda\times 1+1\times K_\lambda)((-1)^{n\dim y}K_\lambda(x)\times y+x\times K_\lambda(y))\\&=(-1)^{n\dim y}K_\lambda(x)\times K_\lambda(y)+x\times K_\lambda(K_\lambda(y))\\&\quad+(-1)^{2n\dim y}K_\lambda{}^2(x)\times y+(-1)^{n\dim K_\lambda(y)}K_\lambda(x)\times K_\lambda(y).\end{aligned}$$

But $\dim K_\lambda(y)=\dim y+n$. So $n\dim K_\lambda(y)=n\dim y+n^2$. Since n^2 is odd, we have $(-1)^n\dim K_\lambda(y)=-(-1)^{n\dim y}$. So the $K_\lambda(x)\times K_\lambda(y)$ terms cancel.

Thus

$$(K_\lambda\times 1+1\times K_\lambda)^2(x\times y)=(K_\lambda{}^2\times 1+1\times K_\lambda{}^2)(x\times y).$$

Before we prove the main theorems in this section, we study some properties of $H_*(A\,;Z_p)$ and K_λ.

Lemma 4. *The following diagram commutative*:

$$\begin{array}{ccccc}
H_0(A)\otimes H_n(S^n) & \xrightarrow{\bar{\mu}} & H_n(A\times S^n) & \xrightarrow{\phi_*} & H_n(X)\\
\downarrow{\scriptstyle q\otimes 1} & & \downarrow{\scriptstyle q_1} & \phi_*\otimes 1 & \searrow{\scriptstyle q_2}\\
H_0(A)\otimes Z_p\otimes H_n(S^n) & & H_n(A\times S^n)\otimes Z_p & \longrightarrow & H_n(X)\otimes Z_p\\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2}\\
H_0(A\,;Z_p)\otimes H_n(S^n) & \xrightarrow{\bar{\mu}} & H_n(A\times S^n\,;Z_p) & \longrightarrow & H_n(X\,;Z_p)
\end{array}$$

where $\bar{\mu}$ is the map in the Künneth formula, q, q_1 and q_2 are maps tensored with Z_p and μ, μ_1 and μ_2 are the maps in the universal coefficient *theorem.*

Proof. $\mu_1 q_1 \bar{\mu}(\{\alpha\} \otimes \{\beta\}) = (\mu_1 q_1)\{\alpha \otimes \beta\} = \mu_1(\{\alpha\otimes\beta\}\otimes 1_p) = \{\alpha\otimes\beta\otimes 1_p\}$ and $\bar{\mu}\mu(q\otimes 1)(\{\alpha\}\otimes\{\beta\}) = \bar{\mu}\mu(\{\alpha\}\otimes 1_p \otimes \{\beta\}) = \bar{\mu}(\{\alpha\otimes 1_p\}\otimes\{\beta\}) = \{\alpha\otimes\beta\otimes 1_p\}$ for $\{\alpha\}\in H_0(A)$, $\{\beta\}\in H_n(S^n)$ and $1_p \in Z_p$ is a generator. So the rectangle on the left is commutative. Also, the rectangles on the right are commutative, because μ_1 and μ_2 are functorial.

We shall denote $1_p = \mu(1\otimes 1_p) \in H_*(A\,;\,Z_p)$ where 1 is the generator of $H_0(A)$ and $1_p \in Z_p$.

Definition 5. Suppose $0 \neq x \in H_i(A\,;\,Z_p)$. We say that x has *λ-depth* d [2] if there is $y \in H_{i-dn}(A\,;\,Z_p)$ such that $K_\lambda^d(y) = x$ and $K_\lambda^{d+1}(z) \neq x$ for any $z \in H_*(A\,;\,Z_p)$.

Since $K_\lambda^0(x) = r_*\phi_*(x\times 1) = r_*\phi_* i_{1*}(x) = r_* f_*(x) = x$, every $x \in H_i(A\,;\,Z_p)$ has a nonnegative λ-depth.

Lemma 6. *Suppose $K_\lambda(1_p) \neq 0$ and $\dim\lambda (=n)$ is odd. If $K_\lambda(x) = 0$, then x has odd λ-depth.*

Proof. Suppose x has λ-depth d and $K_\lambda^d(y) = x$. Then $0 = \Delta_*(K_\lambda(x)) = \Delta_*(K_\lambda^{d+1}(y))$. Let us assume that d is even and that

$$\Delta_*(y) = y\times 1_p + 1_p \times y + \sum_i y_i \times y_i'.$$

Then by Remmark 1,

$$\begin{aligned}\Delta_*(K_\lambda^{d+1}(y)) &= (1\times K_\lambda + K_\lambda\times 1)(K_\lambda^2\times 1 + 1\times K_\lambda^2)^{\frac{d}{2}}(y\times 1_p)\\ &+ (1\times K_\lambda + K_\lambda\times 1)(K_\lambda^2\times 1 + 1\times K_\lambda^2)^{\frac{d}{2}}(1_p\times y)\\ &+ \sum(1\times K_\lambda + K_\lambda + 1)(K_\lambda^2\times 1 + 1\times K_\lambda^2)^{\frac{d}{2}}(y_i\times y_i')\ldots(*).\end{aligned}$$

Since $K_\lambda^d(y) \neq 0$ and $K_\lambda(1_p) \neq 0$ and $H_*(A\,;\,Z_p)$ is a free module, $K_\lambda^d(y)\otimes K_\lambda(1_p) \neq 0$. So $\mu(K_\lambda^d(y)\otimes K_\lambda(1_p)) = K_\lambda^d(y)\times K_\lambda(1_p) \neq 0$. But $K_\lambda^d(y)\times K_\lambda(1_p)$ appears in terms of $\Delta_*(K_\lambda^{d+1}(y))$. Thus $K_\lambda^d(y)\times K_\lambda(1_p) + \sum z_j \times z_j' = 0$ where z_j and z_j' are in formula (*) such that $\dim z_j' = n$ and $\dim z_j = \dim K_\lambda^d(y)$. But since $\dim z_j = \dim K_\lambda^d(y)$ and $\dim y_i < \dim y$, z_j is of the form $K_\lambda^{d+1}(v_j)$ where v_j is some y_i with $\dim v_j = \dim y - n$. Now $\sum K_\lambda^{d+1}(v_j)\times z_j' = -K_\lambda^d(y)\times K_\lambda(1_p)$. Since $H_*(A\,;\,Z_p)$ is a free module, $K_\lambda^d(y)$ is a linear combination of $K_\lambda^{d+1}(v_j)$. i. e, $K_\lambda^d(y) = \sum\alpha_j K_\lambda^{d+1}(v_j) = K_\lambda^{d+1}(\sum\alpha_j v_j)$, for $\alpha_j \in Z_p$. Let $\sum\alpha_j v_j = z$. Then $x = K_\lambda^d(y) = K_\lambda^{d+1}(z)$. So x has λ-depth greater than d, a contradiction.

LEMMA 7. *$H_q(A\,;Z_p)$ (vector space over Z_p) can be written as the direct sum of spaces $A_q{}^d\oplus\cdots\cdots\oplus A_q{}^0$ such that $x\in A_q{}^r$ has λ-depth r.*

Proof. Define $A_q{}^r$ as following; $K_\lambda(H_{q-n}(A\,;Z_p))$ is a subspace of $H_q(A\,;Z_p)$. Let Q_0 be the complementary subspace of $K_\lambda(H_{q-n}(A\,;Z_p))$, that is,

$$K_\lambda(H_{q-n}(A\,;Z_p))\oplus Q_0=H_q(A\,;Z_p).$$

Let $A_q{}^0=Q_0$. $K_\lambda K_\lambda(H_{q-2n}(A\,;Z_p))$ is a subspace of $K_\lambda(H_{q-n}(A\,;Z_p))$. Thus it is a subspace of $H_q(A\,;Z_p)$. Let Q_1 be the complementary subspace of $K_\lambda K_\lambda(H_{q-2n}(A\,;Z_p))$ in $K_\lambda(H_{q-n}(A\,;Z_p))$, that is,

$$K_\lambda K_\lambda(H_{q-2n}(A\,;Z_p))\oplus Q_1=K_\lambda(H_{q-n}(A\,;Z_p))$$

Let $A_q{}^1=Q_1$. Then

$$H_q(A\,;Z_p)=K_\lambda K_\lambda(H_{q-n}(A\,;Z_p))\oplus A_q{}^1\oplus A_q{}^0.$$

Inductively, we define $A_q{}^r$ by the complementary subspace of $K_\lambda{}^{r+1}(H_{q-(r+1)n}(A\,;Z_p))$ in $K_\lambda{}^r(H_{q-rn}(A\,;Z_p))$, that is,

$$K_\lambda{}^{r+1}({}_{-(r}H_{q+1)n}(A\,;Z_p))\oplus A_q{}^r=K_\lambda{}^r(H_{q-rn}(A\,;Z_p)).$$

Let $\left[\dfrac{q}{n}\right]=d$, where $[\ \ ]$ is the Gauss function. Then we have

$$K_\lambda{}^d(H_{q-dn}(A\,;Z_p))\oplus A_q{}^{d-1}\oplus\cdots\oplus A_q{}^0=H_q(A\,;Z_p).$$

But since $q-(d+1)n<0$, $H_{q-(d+1)n}(A\,;Z_p)=0$. So $K_\lambda{}^{d+1}(H_{q-(d+1)n}(A\,;Z_p)=0$ and therefore $K_\lambda{}^d(H_{q-dn}(A\,;Z_p))=Q_d=A_q{}^d$. Consequently,

$$H_q(A\,;Z_p)=A_q{}^0\oplus A_q{}^1\oplus\cdots\cdots\oplus A_q{}^d,$$

where $d=\left[\dfrac{q}{n}\right]$. If $x\in A_q{}^r$, then $x\in K_\lambda{}^r(H_{q-rn}(A\,;Z_p))$ and $x\in Q_r$. So there is a $y\in H_{q-rn}(A\,;Z_p)$ such that $K_\lambda{}^r(y)=x$ and $x\neq K_\lambda{}^{r+1}(z)$ for any $z\in H_*(A\,;Z_p)$. Thus x has λ-depth r.

In particular, we have $K_\lambda(A_q{}^r)\supset A_{q+n}^{r+1}$. In fact,

$$\begin{aligned}K_\lambda{}^{r+2}(H_{q+n-(r+2)n}(A\,;Z_p))\oplus A_{q+1}^{r+1}&=K_\lambda{}^{r+1}(H_{q+n-(r+1)n}(A\,;Z_p))\\&=K_\lambda{}^{r+1}(H_{q-rn}(A\,;Z_p))=K_\lambda(K_\lambda{}^r(H_{p-rn}(A\,;Z_p))\\&=K_\lambda(K_\lambda{}^{r+1}(H_{q-(r+1)n}(A\,;Z_p)\oplus A_q{}^r))\\&\subset K_\lambda{}^{r+2}(H_{q+n-(r+2)n}(A\,;Z_p))+K_\lambda(A_q{}^r)).\end{aligned}$$

But since A_{q+n}^{r+1} is complementary to $K_\lambda{}^{r+2}(H_{q+n-(r+2)n}(A\,;Z_p))$, $A_{q+n}^{r+1}\subset K_\lambda(A_q{}^r)$.

LEMMA 8. *Let $A_q{}^r$ be the subspace of $H_q(A\,;Z_p)$ defined in Lemma 7. If d is even, K_λ: $A_q{}^d\cong A_{q+n}^{d+1}$.*

Proof. We first show that $K_\lambda(A_q{}^d)=A^{d+1}_{q+n}$. For suppose not. Then there exists an $x\in A_q{}^d$ such that $K_\lambda(x)$ has λ-depth greater than $d+1$. Thus there is a $z\in H_*(A\ ;\ Z_p)$ such that $K_\lambda(x)=K_\lambda{}^r(z)$ for $r>d+1$. Let $y=K_\lambda{}^{r-1}(z)$. Then $K_\lambda(x)=K_\lambda(y)$. Since $x\in A_q{}^d$, $x=K_\lambda{}^d(z')$ for some $z'\in H_{q-dn}(A\ ;\ Z_p)$ and $x\neq K_\lambda{}^{d+1}(w)$ for any $w\in H_*(A\ ;\ Z_p)$. Then

$x-y=K_\lambda{}^d(z')-K_\lambda{}^{r-1}(z)=K_\lambda{}^d(z'-K_\lambda{}^{r-d-1}(z))$ and $x-y\neq K_\lambda{}^{d+1}(w)$ for any $w\in H_*(A\ ;\ Z_p)$, because if $x-y=K_\lambda{}^{d+1}(w)$, $x=K_\lambda{}^{d+1}(w)+K_\lambda{}^{r-1}(z)=K_\lambda{}^{d+1}(w+K_\lambda{}^{r-d-2}(z))$ which is a contradiction. But since $K_\lambda(x-y)=0$ and d is even, $x-y=0$ by Lemma 6. Thus y has λ-depth d. This is a contradiction to $y=K_\lambda{}^{r-1}(z)$, where $r-1>d$. Consequently,

$$K_\lambda(A_q{}^d)=A^{d+1}_{q+n}.$$

Similarly, if $K_\lambda(x-y)=0$, then $x-y=0$. This implies that K_λ is injective.

Let $h:\Pi_n(x)\to H_n(X\ ;\ Z)$ be the Hurewicz homomorphism. We shall define $h_p:\Pi_n(X)\to H_n(X\ ;\ Z)\to H_n(X\ ;\ Z_p)$ as composition of h tensored with Z_p. h_p will be called *the mod p Hurewicz homomorphism.* We shall let h_∞ stand for the Hurewicz map $h_\infty:\Pi_n(X)\to H_n(X\ ;\ Q)$ where Q is the rational field.

THEOREM 9. *Let X and A be topological spaces and A have a finitely generated integer homology and $f:A\to X$ be a map which has a left homotopy inverse r. If n is an odd integer, then $G_n{}^f(X,A,x_0)$ is contained in the kernel of r_*h_p, for any prime number p or ∞ provided $\chi(A)\neq 0$.*

Proof. Suppose $\alpha\in G_n{}^f(X,A,x_0)$ is not contained in the kernel of r_*h_p. Then, if $\phi:A\times S_n\to X$ is affiliated to α with trace g, we have,

$$\begin{aligned}0\neq r_*h_p(\alpha)&=r_*\mu_2q_2\phi_*i_*(\lambda)=r_*\mu_2q_2\phi_*(\mu(1\otimes\lambda))\\&=r_*\phi_*(\bar\mu(\mu(1\otimes 1_p)\otimes\lambda)=r_*\phi_*(1_p\times\lambda)=K_\lambda(1_p),\end{aligned}$$

by Lemma 4. Thus if $K_\lambda(x)=0$, then x has odd λ-depth by Lemma 6. This fact implies that $K_\lambda:A_q{}^{2r}\cong A_q{}^{2r+1}$, for $r\geq 0$, by Lemma 8. Let $\chi(H_*(A\ ;\ Z_p))=\sum_i(-1)^i(\dim H_i(A\ ;\ Z_p))$. Since $H_*(A\ ;\ Z_p)$ is finitely generated. $\chi(H_*(A\ ;\ Z_p))$ is well-defined. But

$$\begin{aligned}\chi(H_*(A\,;\,Z_p)) &= \sum_q (-1)^q(\dim H_q(A\,;\,Z_p))\\ &= \sum_q (-1)^q(\sum_r \dim A_q^r) \quad \text{by Lemma 7}\\ &= \sum_q (-1)^q \sum_r (\dim A_q^{2r} + (-1)^n \dim A_{q+n}^{2r+1}).\end{aligned}$$

Since $\dim A_q^{2r} \cong \dim A_{q+n}^{2r+1}$ and n is odd, $\chi(H_*(A\,;\,Z_p))=0$. But $\chi(H_*(A\,;\,Z_p))=\chi(A)$, see [2]. This is a contradiction to hypothesis.

LEMMA 10. *Suppose* $K_\lambda(1_p)\neq 0$ *and dim* $\lambda(=n)$ *is even. If* $K_\lambda(x)=0$, *then* λ*-depth* d *of* x *is equal to* -1 *mod* p.

Proof. Suppose $K_\lambda{}^d(y)=x$. Then since n is even,

$$0=\Delta_*(K_\lambda(x))=\Delta_*(K_\lambda{}^{d+1}(y))=(\sum_i \binom{d+1}{i} K_\lambda{}^i \times K_\lambda{}^{d+1-i})(\Delta_*(y))$$

by Remark 1. Now $\Delta_*(y)=y\times 1_p+1_p\times y+\sum y_i\times y_i'$. Thus $(d+1)$ $K_\lambda{}^d(y)\times K_\lambda(1_p)$ must appear in $\Delta_*(K_\lambda{}^{d+1}(y))$. Now since $\Delta_*(K_\lambda{}^{d+1}(y))=0$, $H_*(A\,;\,Z_p)$ is a free module, $K_\lambda{}^d(y)\neq 0$ and $K_\lambda(1_p)\neq 0$,

$$(d+1)K_\lambda{}^d(y)\times K_\lambda(1_p)+\sum z_i\times z_i'=0,$$

where $\dim z_i'=n$ and $\dim z_i=\dim K_\lambda{}^d(y)$. Thus $\sum z_i\times z_i'$ must come from terms of the form $(K_\lambda{}^{d+1}\times 1)(\sum v_i\times z_i')$ where $\sum v_i\times z_i'$ is the sum of all terms in $\Delta_*(y)$ with the z_i' having dimension n. Thus

$$(d+1)K_\lambda{}^d(y)\times K_\lambda(1_p)+\sum K_\lambda{}^{d+1}(v_i)\times z_i'=0$$

Therefore, $(d+1)K_\lambda{}^d(y)=\sum \alpha_i K_\lambda{}^{d+1}(v_i)$, where $\alpha_i\in Z_p$. Let $z=\alpha_i v_i$. Then if $(d+1)\neq 0 \pmod p$, $x=K_\lambda{}^d(y)=(d+1)^{-1}K_\lambda{}^{d+1}(z)$. This is a contradiction to the fact that x has λ-depth d. So $d\equiv -1 \pmod p$.

THEOREM 11. *Let* X *be a topological space and* A *be a topological space with finitely generated integer homology and* $f: A\to X$ *be a map with left homotopy inverse* r. *Suppose* p *is a prime number which does not divide* $\chi(A)$. *Then* $G_n{}^f(X, A, x_0)\subset$ *kernel* r_*h_p *for even* n.

Proof. Assume that $\alpha\in G_n{}^f(X, A, x_0)$ is not in the kernel of r_*h_p. Now suppose that $\phi: A\times S^n\to X$ is affiliated to α. Then $K_\lambda(1_p)\neq 0\in H_n(A\,;\,Z_p)$ by Lemma 4 and definition of K_λ. By Lemma 7, $H_q(A\,;\,Z_p)=A_q{}^0\oplus\cdots\cdots\oplus A_q{}^r$ such that $K_\lambda(A_q{}^d)\supset A_{q+n}^{d+1}$. Now, as in the proof of the Lemma 8, $K_\lambda: A_q{}^d\cong A_{q+n}^{d+1}$ if $d+1$ is not a multiple of p.

$$\begin{aligned}\chi(H_*(A\,;\,Z_p)) &= \sum_q (-1)^q \dim H_q(A\,;\,Z_p)\\ &= \sum_q (-1)^q(\sum_d \dim A_q{}^d)\\ &= \sum_q \{(-1)^q \dim A_q{}^0 + (-1)^{q+n} A_{q+n}^1 + \cdots\end{aligned}$$

$$\cdots(-1)^{q+dn}\dim A^d_{q+dn}+\cdots\}$$
$$=\sum_q(-1)^q(\sum_{d=0}\dim A^d_{q+dn}). \quad \text{(for } n \text{ is even)}$$

Since $\dim A_q^{kp}=\dim A^{kp+1}_{q+n}=\cdots\cdots=\dim A^{kp+p-1}_{q+(p-1)n}$, we see that $\sum_d \dim A^d_{q+dn}$ is a multiple of p and so $\chi(H_*(A\,;Z_p))$ is a multiple of p.

Corollary 12. *Let X and A be topological spaces and $f: A\to X$ has a left homotopy inverse r. If A has finitely generated integer homology and $\chi(A)=1$, then $G_n^f(X, A, x_0)\subset Ker\ r_*h_p$ for all n and prime p.*

Recall that there is a transformation $k: \Pi_n(X, A, *)\to H_n(X, A)$ called the Hurewicz homomorphism [4]. Consider a map

$$H: (X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\to(X, A, x_0)$$

such that $H(x, u)=x$, when $x\in X$ and $u\in J^{n-1}$. Then the map f: $(I^n, \partial I^n, J^{n-1})\to(X, A, x_0)$ defined by $f(u)=H(x_0, u)$, where x_0 is a base point of X, represents an element $\alpha=[f]\in\Pi_n(X, A, x_0)$. The set of all elements $\alpha\in\Pi_n(X, A, x_0)$ obtained in the above manner from some H will be denoted by $G_n^{Rel}(X, A, x_0)$ ([6]).

Theorem 13. *Let A be the retract of CW-complex X. Then $G_n(X, A, x_0)\subset ker\ r_*h_p$ and $G_n^{Rel}(X, A, x_0)\subset ker\ k_p$ if and only if $G_n(X, A, x_0)\subset ker\ h_p$, where r is a retraction and h_p and k_p are Hurewicz homomorphisms tensored with Z_p for all prime number p.*

Proof. Consider the following commutative diagram of exact sequences;

$$\begin{array}{ccccccccc}
\longrightarrow G_{n+1}^{Rel}(X, A) & \longrightarrow & G_n(A) & \underset{r_*}{\overset{i_*}{\rightleftarrows}} & G_n(X, A) & \overset{j_*}{\longrightarrow} & G_n^{Rel}(X, A) & \longrightarrow & G_{n-1}(A)\longrightarrow \\
\downarrow k_p & & \downarrow h_p & & \downarrow h_p & & \downarrow k_p & & \downarrow h_p \\
\to H_{n+1}(X, A\,;Z_p) & \to & H_n(A\,;Z_p) & \underset{r_*}{\overset{i_*}{\rightleftarrows}} & H_n(X\,;Z_p) & \overset{j_*}{\to} & H_n(X, A\,;Z_p) & \to & H_{n-1}(A\,;Z_p)\to
\end{array}$$

Since j_* is surjective, the sufficient condition is trivial.

Conversely, suppose $G_n^{Rel}(X, A, x_0)\subset\ker k_p$ and $G_n(X, A, x_0)\subset\ker r_*h_p$. Then $j_*h_p(G_n(X, A, x_0)) = k_pj_*(G_n^{Rel}(X, A, x_0)) = k_p(G_n^{Rel}(X, A, x_0)) =0$. Thus $h_p(G_n(X, A, x_0))\subset\ker j_*=\mathrm{Im}\ i_*$. So, for every $\alpha\in G_n(X, A, x_0)$, there is a $\beta\in H_n(A\,;Z_p)$ such that $i_*(\beta)=h_p(\alpha)$. But

$$\beta=r_*i_*(\beta)=r_*h_p(\alpha)=0.$$

Hence $h_p(\alpha)=i_*(\beta)=0$. Consequently $G_n(X, A)\subset\ker h_p$.

COROLLARY 14. *Let A be a retract of X and have a finitely generated integer homology group. Let n be an odd integer and Euler–poincare number $\chi(A)\neq 0$. Then $G_n(X, A, *)\subset ker\, h_p$ if and only if $G_n^{Rel}(X, A, *)\subset ker\, k_p$.*

COROLLARY 15. *Let A be a retract of X and have a finitely generated homology group. Suppose p is a prime which does not divided $\chi(A)$. Then $G_n(X, A, *)$ is contained in the kernel of h_p if and only if $G_n^{Rel}(X, A, *)$ is contained in the kernel of k_p for even n.*

Thus the condition that $G_n^{Rel}(X, A, *)\subset\ker k_p$ is the minimum condition so that $G_n(X, A, *)\subset\ker h_p$ under the above conditions. Moreover, Corollary 14 and Corollary 15 are a generalization of theorems proved by Gottlieb because $G_n(X, X, *)=G_n(X)$ and $G_n^{Rel}(X, X, *)=0$.

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