J. Korean Math. Soc. 25(1988), No. 2, pp. 325-332

UNIFORMLY ALMOST RECURRENCE IN DYNAMICAL SYSTEMS

Ki-Shik Koo

1. Introduction

There are many recursive concepts in dynamical systems and those concepts play an important role in the stability theory of dynamical systems. Our object in this paper is to analyze the flows in the vicinity of an almost recurrent point. The concept of uniformly almost recurrence is introduced. For example, a Liapunov stable almost recurrent point is uniformly almost recurrent. Some dynamical properties of uniformly almost recurrent points are obtained. If a flow (X, π) has a unique minimal set and X is compact, then this flow contains one uniformly almost recurrent trajectory closure. Also it is shown that for an uniformly almost recurrent point $x \in X$ any two trajectory closures in $J(x)$ must intersect.

2. Definitions and notation

Throughout this paper, we shall assume (X, π) is a given flow on a locally compact metric space with a metric d. R, R^+, R^- will denote the reals, nonnegative reals, and nonpositive reals, respectively. The ε-neighborhood of $x \in X$ is denoted by $B(x, \varepsilon)$. For a point $x \in X$ and $t \in R$, $\pi(x, t)$ is denoted by xt for brevity. The trajectory, trajectory closure, limit, prolongation, prolongational limit relations are denoted, respectively, by C, K, L, D, and J, respectively, with the unilateral versions of these relations carrying the appropriate superscript $+$ or $-$.

We say that the (positive) trajectory $(C^+(x))$ $C(x)$ uniformly approximates the set $Q \subset X$ if for any $\varepsilon > 0$ there exists a number $T > 0$ such that $Q \subset B(xt[0, T], \varepsilon)$ for any t in $(R^+)R$. A point x of X

Received August 31, 1987.

is said to be almost recurrent if the trajectory $C(x)$ uniformly approximates the set $\{x\}$. It is well known that the trajectory closure of an almost recurrent point is a minimal set.

Here we introduce an uniformly almost recurrent point.

DEFINITION. A point x of X is called uniformly almost recurrent if, given $\varepsilon>0$, there exists a $\delta>0$ and a number $T>0$ such that $yt[0, T] \cap B(x, \varepsilon) \neq \phi$ for any $y \in B(x, \delta)$ and $t \in R$.

A point x of X is said to be Liapunov stable if for any $\varepsilon>0$ there exists a $\delta>0$ such that for any $y \in B(x, \delta)$ we have $d(xt, yt)<\varepsilon$ for any $t \in R$. A point $x \in X$ is said to be positively (negatively) Laglange stable if $L^+(x)$ $(L^-(x))$ is a nonempty compact set. A point x of X is said to be Laglange stable if it is positively and negatively Laglange stable.

A set $\{r\}$ of real numbers is said to be relatively dense if there exists a number $L>0$ such that at least one point of the set $\{r\}$ is in any segment $[l, l+L]$ for any $l \in R$.

3. Basic results

PROPOSITION 3.1. *Let a point x of X be Liapunov stable and almost recurrent. Then x is uniformly almost recurrent.*

Proof. Let $\varepsilon>0$ be given. It is easily shown that if $x \in X$ is almost recurrent, then there is a relatively dense set $\{r\}$ with a number $L>0$ such that $d(x, xr_n) < \frac{1}{2}\varepsilon$ for any r_n in $\{r\}$. Moreover, since x is Liapunov stable there exists a $\delta>0$ corresponding to ε such that for any $y \in B(x, \delta)$ $d(xt, yt) < \frac{1}{2}\varepsilon$ for each $t \in R$. Therefore we have $d(x, yr_n)<\varepsilon$ for any $y \in B(x, \delta)$ and $r_n \in \{r\}$. This implies that $yt[0, L] \cap B(x, \varepsilon) \neq \phi$ for any $t \in R$ and $y \in B(x, \delta)$. Thus we conclude that x is uniformly almost recurrent.

THEOREM 3.2. *A necessary and sufficient condition for a point x of X to be uniformly almost recurrent is that for any sequence $\{x_n t_n\}$ with $x_n \to x$ and $\{t_n\} \subset R$ the sequence $\{x_n t_n\}$ has a convergent subsequence and* *for any neighborhood U of x we have $J(x) \subset U[0, T]$ for some $T \in R^+$.*

Proof. (Necessity) Suppose that $x \in X$ is uniformly almost recurrent. First, let U be a neighborhood of x. Select an $\varepsilon > 0$ with $\overline{B(x, \varepsilon)} \subset U$. Then there exists a $\delta > 0$, a number $T > 0$ corresponding to ε such that $yt[-T, 0] \cap B(x, \varepsilon) \neq \phi$ for each $y \in B(x, \delta)$ and $t \in R$ so that

(1) $\quad C(B(x, \delta)) \subset B(x, \varepsilon)[0, T] \subset \overline{B(x, \varepsilon)}[0, T] \subset U[0, T]$.

Moreover, y belongs to $J(x)$ means there exists a sequence $\{x_n\}$ in X, $\{t_n\}$ in R such that $x_n \to x$, $t_n \to \pm\infty$, and $x_n t_n \to y$. By (1) we may assume that $\{x_n t_n\} \subset \overline{B(x, \varepsilon)}[0, T]$ and therefore we have $y \in \overline{B(x, \varepsilon)}[0, T] \subset U[0, T]$. Consequently, we have $J(x) \subset U[0, T]$.

On the other hand, let $\{x_n t_n\}$ be any sequence with $x_n \to x$. Then we may assume that $\{x_n t_n\}$ is in $\overline{B(x, \varepsilon)}[0, T]$ and $\overline{B(x, \varepsilon)}$ is compact. Thus the sequence $\{x_n t_n\}$ has a convergent subsequence.

(Sufficiency) Let suppose that x of X is not uniformly almost recurrent. Then there exists an $\varepsilon > 0$, a sequence of intervals $\{[t_n, r_n]\}$ such that $x_n \to x$, $r_n - t_n \to +\infty$, and $d(x, x_n[t_n, r_n]) \geqslant \varepsilon$. Let $t_n' = \frac{t_n + r_n}{2}$. We now show that $\{t_n'\}$ is unbounded. Without loss of generallity assume that $\{t_n'\}$ converges to some $t_0 \in R$. Consider a sequence of points $\{x_n'\}$, where $x_n' = x_n t_n'$. Then $x_n' \to x t_0$. By the continuity of π there exists a $\delta > 0$ such that $d(w, x t_0) < \delta$ implies $d(w(-t_0), x) < \varepsilon$. Since $r_n - t_n \to +\infty$ we can choose a number N large enough that $d(x_N', x t_0) < \delta$ and $r_N - t_N' > |t_0|$. Thus we have

(2) $$d(x_N'(-t_0), x) < \varepsilon.$$

But since

$$x_N'(-t_0) = x_N\left(\frac{t_N + r_N}{2} + (-t_0)\right) \in x_N[r_N, t_N]$$

we have $d(x_N'(-t_0), x) \geqslant \varepsilon$. This contradicts to (2). Consequently, $\{t_n'\}$ is unbounded.

By assumption without loss of generality we may assume that $x_n t_n' \to y \in X$. Then $y \in J(x)$ and $J(x) \subset B(x, \varepsilon)[0, T]$ for some $T > 0$. This implies that there exists a $s \in [-T, 0]$ with $ys \in B(x, \varepsilon)$. Let $d(x, ys) = \eta$. Consider a neighborhood $B(ys, \sigma)$ of ys, where $\eta + \sigma < \varepsilon$. By the continuity of π there exists a $\nu > 0$ such that $d(z, y) < \nu$ implies that $d(zs, ys) < \sigma$. Choose a number P large enough that $d(y, x_P') < \nu$ and $\frac{t_P' - t_P}{2} > |s|$. Then we have

$$x_P's = x_P\left(\frac{t_P + r_P}{2} + s\right) \in x_P[t_P, r_P].$$

This shows that

(3) $$d(x, x_P's) \geq \varepsilon.$$

On the other hand, since $d(y, x_P') < \nu$ we have

$$d(x, x_P's) \leqslant d(x, ys) + d(ys, x_P's) < \eta + \sigma < \varepsilon.$$

But this contradicts to (3). Therefore the point x is uniformly almost recurrent and the proof of the theorem is complete.

COROLLARY 3.3. *Let a point $x \in X$ be uniformly almost recurrent. Then x has a neighborhood such that any point in it has nonempty compact connected prolongation and prolongational limit set.*

Proof. The arguments in the proof of Theorem 3.2 show that for a neighborhood U of x with compact closure we can select a neighborhood $B(x, \delta)$ satisfying that $C(B(x, \delta)) \subset \bar{U}[0, T]$. By the definitions of $J(y)$ and $D(y)$ we have $J(y), D(y) \subset \bar{U}[0, T]$. Since $J(y)$ and $D(y)$ are closed they are compact. It has proved that they are connected whenever they are compact [2. p. 26].

LEMMA 3.4. *The set of uniformly almost recurrent points are invariant.*

Proof. The proof is straightforward and is thus omitted.

LEMMA 3.5. *Every point in the limit set of an uniformly almost recurrent point is also uniformly almost recurrent.*

Proof. Let x of X be uniformly almost recurrent and $y \in L(x)$. We shall show that y is uniformly almost recurrent by using Theorem 3.2. Let U be an arbitrary neighborhood of y. Since $y \in L(x)$ there exists a point x' in $C(x)$ and a neighborhood U' of x' such that $x' \in U' \subset U$. Since the trajectory closure of x is compact minimal we have $L^+(y) = L^+(x) = L^-(y) = L^-(x) = K(x) = K(y)$ so that $J(x) = J(x') = J(y)$. By Theorem 3.2 and Lemma 3.4 $J(x') \subset U'[0, T]$ for some number $T > 0$. Therefore $J(y) \subset U'[0, T] \subset U[0, T]$.

On the other hand, for any sequence $\{y_n t_n\}$ with $y_n \to y$ we shall show that this sequence has a convergent subsequence. If the set $\{t_n\}$ is bounded, then $\{y_n t_n\}$ converges. Hence we are done. So we assume that $t_n \to +\infty$. If $t_n \to -\infty$. then the proof is similar. Consider neighborhoods $B\left(x, \frac{1}{k}\right)$ for each natural number k. Since $x \in L^+(y)$, for each $k \in N$, there exists a neighborhood W_k of x and a number $s_k > 0$ such that $W_k s_k \subset B\left(x, \frac{1}{k}\right)$. Thus we can choose a subsequence $\{y_{n_k} t_{n_k}\}$ of $\{y_n t_n\}$ with $y_{n_k} s_k \in B\left(x, \frac{1}{k}\right)$ and $t_{n_k} > s_k$. Since $y_{n_k} t_{n_k} = (y_{n_k} s_{n_k})(t_{n_k} - s_{n_k})$ with $y_{n_k} s_{n_k} \to x$ and x is uniformly almost recurrent we have the sequence $\{y_{n_k} t_{n_k}\}$ has a convergent subsequence. Consequently, we conclude that y is uniformly almost recurrent and the proof is complete.

THEOREM 3.6. *All points in the closure of the trajectory of an uniformly almost recurrent point are also uniformly almost recurrent.*

Proof. This follows from Lemma 3.4 and Lemma 3.5.

It is an interesting problem in dynamical system theory whether the dynamical properties of a point can be inherited to the points in its dynamical limit sets. For almost recurrent points all points in the trajectory closure of an almost recurrent point are almost recurrent. Above theorem shows that the trajectory closure of an uniformly almost recurrent point is a compact minimal set consisting of uniformly almost recurrent points. However, uniformly almost recurrent point need not have a prolongational limt set each of whose point is uniformly almost recurrent. Moreover, even if all points in the limit set of x of X is uniformly almost recurrent x need not be uniformly almost recurrent. For example, as the space X take the one circle. We stipulate an arbitrary point x to be rest point and the set $X - \{x\}$ to be one regular trajectory. It is easily shown that x is uniformly almost recurrent. For a point y in $X - \{x\}$ we have $L(y) = \{x\}$ so that $y \in J(x)$. But y is not uniformly almost recurrent.

LEMMA 3.7. *Let X be a compact space and contain only one minimal set. Then it contains an unique uniformly almost recurrent trajectory closure.*

Proof. Let M be an unique minimal set in X and $x \in M$. Then M is the trajectory closure of the point $x \in X$. We shall proceed by showing that the point x is uniformly almost recurrent. For each point $y \in X$

each $L^+(y)$ and $L^-(y)$ is a nonempty compact invariant set and hence contains minimal set M. Thus $x \in L^+(y) \cap L^-(y)$ for each point $y \in X$. Consider neighborhoods U and W of x with $\bar{U} \subset W$ and $\bar{U}$ compact. For each point $y \in X$ define $T_y = \inf\{t \in R^+ : yt \in U\}$. Since $x \in L^+(y)$ T_y is welldefined. Also define $T = \sup\{T_y : y \in \bar{U}\}$. We now show that T is bounded. By the continuity of π, for every point z in $\bar{U}$ there exists an open neighborhood V_z of z with $V_z s \subset U$ for some s between T_z and T_z+1. Note that for any point p in $\bar{U}$ $T_p \leqslant T_z+1$ for all point p in V_z. The set $\{V_z : z \in \bar{U}\}$ is an open covering of $\bar{U}$ and hence contains a finite subcovering $\{V_{z_i} : i=1, \cdots, n\}$. Therefore for everypoint p in $\cup\{V_{z_i} : i=1, \cdots, n\}$ we have $T_p \leqslant \max\{T_{z_i}+1 : i=1, \cdots, n\}$. This shows that T is bounded. Let $\sigma_q = \sup\{t \in R^- : qt \in \bar{U}\}$ for a point q in X. Since $x \in L^-(q)$ for each $q \in X$ σ_q is well defined for all points q in X. Therefore for any point $q \in X$ we have

$$q \in q\sigma_q[0, T] \subset \bar{U}[0, T] \subset W[0, T]$$

This implies that $X = W[0, T]$. On the other hand, since $J(x) = X$, we have $J(x) = W[0, T]$. Thus we conclude that the point x is uniformly almost recurrent by Theorem 3.2. The uniqueness of the trajectory closure follows from the minimality of the set M. This completes the proof of the theorem.

THEOREM 3.8. *Let X be a compact minimal set. Then any point in X is uniformly almost recurrent.*

Proof. This follows from Theorem 3.6 and Lemma 3.7.

The following result is due to Nemytskii. For a proof see [2].

LEMMA 3.9. *Let $x \in X$ be positively Laglange stable. Then the set $L^+(x)$ is minimal if and only if the semitrajectory $C^+(x)$ uniformly approximates $L^+(x)$.*

PROPOSITION 3.10. *Let a point $x \in X$ be uniformly almost recurrent. Then $J(x)$ is minimal if and only if the trajectory $C(x)$ uniformly approximates $J(x)$.*

Proof. First, suppose that the trajectory $C(x)$ uniformly approximates $J(x)$ but $J(x)$ is not minimal. Then there are points y and z in

$J(x)$ such that $z \in K(y)$. Let $d(z, K(y))=\varepsilon$. By uniform approximation there is a number $T>0$ such that $J(x) \subset B(xt[0, T], \varepsilon)$ for all $t \in R$. Hence we can find a x' in $C(x)$ with $x' \in B(z, \varepsilon)$. Choose a $\eta>0$ such that $B(x, \eta) \subset B(z, \varepsilon)$. Then there exists a number $L>0$ corresponding to η such that $J(x) \subset B(x', \eta)[0, L]$. Therefore $y \in B(z, \varepsilon)[0, L]$ and $C(y) \cap B(z, \varepsilon) \neq \phi$. We have shown that $d(z, K(y)) < \varepsilon$. But this is a contradiction. Thus $J(x)$ is minimal.

Conversely, assume $J(x)$ is minimal. Then we have $J(x)=L^+(x)=L^-(x)$. By Lemma 3.9 for given $\varepsilon>0$ there are numbers T_1, T_2 in R^+ with $J(x) \subset B(xt[0, T_1], \varepsilon)$ for each $t \in R^+$ and $J(x) \subset B(xt[-T_2, 0], \varepsilon)$ for each $t \in R^-$. Let $T=T_1+T_2$. Then $J(x) \subset B(xt[0, T], \varepsilon)$ for each $t \in R$. This completes the proof of Proposition 3.10.

There are some answers for the question of under which conditions the two trajectory closures $K(x)$ and $K(y)$ intersect if $y \in J(x)$ [3]. For uniformly almost recurrent points we have the following results.

LEMMA 3.11. *Let $x \in X$ be uniformly almost recurrent. Then for any point y in $J(x)$ the trajectory closures $K(x)$ and $K(y)$ must intersect.*

Proof. Suppose that $K(x)$ and $K(y)$ does not intersect. Then we can choose a neighborhood U of x with $U \cap K(y)=\phi$. By Theorem 3.2 there exists a number $T>0$ such that $J(x) \subset U[0, T]$. Hence $y \in U[0, T]$ and this implies $C(y) \cap U \neq \phi$. This is absurd. Hence $K(x)$ and $K(y)$ must intersect.

THEOREM 3.12. *Let x of X be uniformly almost recurrent. Then any two trajectory closures in $J(x)$ intersect.*

Proof. Let $y, z \in J(x)$. By Lemma 3.11 we have $K(x) \cap K(y) \neq \phi$ and $K(x) \cap K(z) \neq \phi$. Since $K(x)$ is minimal $K(x)$ is contained in $K(y)$ and $K(z)$. Therefore $K(y)$ and $K(z)$ intersect.

EXAMPLES. 1) consider a flow (T, π) defined on a torus T by means of the planar differential system

$$\frac{d\phi}{dt}=f(\phi, \theta), \quad \frac{d\theta}{dt}=\alpha f(\phi, \theta) \tag{1}$$

where $f(\phi, \theta) \equiv f(\phi+1, \theta+1) \equiv f(\phi+1, \theta) \equiv f(\phi, \theta+1)$, and $f(\phi, \theta)>0$

if ϕ and θ are not both 0 (mod 1), $f(0, 0)=0$. Let $\alpha>0$ be irrational. Let $p\in X$ be a rest point (see 2.7 in [2]). Since $\{p\}$ is a unique minimal set in T $\{p\}$ is a unique uniformly almost recurrent trajectory closure (cf. Lemma 3.7). The point p is uniformly almost recurrent but not Liapunov stable (cf. Proposition 3.1).

2) In (1) let $f(\phi, \theta)=1$. Then in this flow T is a compact minimal set. Therefore all points in T are uniformly almost recurrent (cf. Theorem 3.8)

References

1. K. Ahmad S. Elaydi, *Nearly periodic dynamical systems*, J. Appli. Analysis, To appear.
2. N. Bhatia and G. Szego, *Stability theory of dynamical systems*, Springer-Verlag, Berlin and New york, (1970).
3. R. Knight, *Minimal sets in recurrent discrete flows*, Proc. Amer. Math. Soc, **100**(1987), 195-198.
4. K. Sibirsky, *Introduction to topological dynamics*, Noordhoff. International Pub., Leyden, (1975).

Daejeon University
Daejeon 300-120, Korea