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THEOREMS AND EXAMPLES FOR *R*-TYPE SUMMABILITY METHODS

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1. Introduction

The concept of $\boldsymbol{R}$-type summability methods (RSMs) was introduced in [1] as an aid in the study of the strong summability fields associated with certain methods. In [2] an RSM was used to help identify the strong convergence field associated with the well known space $bs+c$.

In this paper we adopt the notation of [1]. In particular, *summability method* will simply mean a real valued linear functional S defined on some subspace $c_S \subseteq \omega$, where ω is the linear space of all real sequences. We shall call S *regular* if $c \subseteq c_S$ and $S(x) = \lim x$ for each $x \in c$. We call S *non-negative* if $S(x) \geq 0$ for each $x \in c_S$ with $x \geq 0$ (i. e. $x_i \geq 0$ for all i). Further, we let

$$c_S{}^0 = \{x \in c_S : S(x) = 0\},$$
$$|c_S|^0 = \{x \in \omega : |x| \in c_S{}^0\},$$

and

$$|c_S| = \{x \in \omega : x - r \in |c_S|^0 \text{ for some real } r\}.$$

The sets $|c_S|$ and $|c_S|^0$ are the strong summability fields associated with the method S. Unless S is "nice", however, these may not even be subspaces of ω. In [1] we find the following definition and theorem: a method S will be called an RSM when S is regular and $m \cdot |c_S|^0 = |c_S|^0$ (i. e. $|c_S|^0$ is solid); if S is an RSM, then $|c_S|$ and $|c_S|^0$ are subspaces of c_S and $c_S{}^0$ respectively and, furthermore, $c \subseteq |c_S|$ and $c_0 \subseteq |c_S|^0$.

In section 2 we present some results concerning RSMs with a view to better understanding these methods. For example, it turns out that all RSMs are non-negative and therefore continuous (with respect to a particular topology). We also investigate sufficient conditions for S to be an RSM.

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In sections 3 and 4 we give several interesting examples which clarify the necessity or sufficiency of the various conditions for RSMs. In section 4 we study particularly matrix methods vis-a-vis RSMs.

2. Conditions for RSMs

We start with two propositions, the proofs of which are routine.

PROPOSITION 1. *For any summability method S, the two following conditions are equivalent:*

i) $m \cdot |c_S|^0 \subseteq c_S$;

ii) $x \in |c_S|^0, |y| \leq |x| \Rightarrow y \in c_S$.

Note the condition $m \cdot |c_S|^0 \subseteq c_S$ does not imply that $|c_S|^0$ is a subspace of c_S (see example 3 of section 3).

PROPOSITION 2. *A regular method S is an RSM if and only if the following condition holds:*

$$x \in |c_S|^0, |y| \leq |x| \Rightarrow y \in |c_S|^0.$$

Proof. The condition is just another way of stating that $|c_S|^0$ is solid.

PROPOSITION 3. *If S is an RSM, then S is nonnegative.*

Proof. Let $x \in c_S$, $x \geq 0$ and assume that $S(x) = -r$ where $r > 0$. Then $x + r \geq 0$ and so (with $e = (1, 1, 1, \ldots\ldots)$)

$$S(|x+r|) = S(x+r) = S(x) + S(re) = -r + r = 0.$$

Hence $x + r \in |c_S|^0$. Since $0 \leq re \leq x + r$, we have, by proposition 2, that $re \in |c_S|^0$. Hence $S(re) = 0$ which is a contradiction.

We now define the *uniform topology* on ω. For any $x \in \omega$ and $\varepsilon > 0$, let

$$N_\varepsilon(x) = \{y : \sup\{|x_i - y_i| : i = 1, 2, 3, \ldots\} < \varepsilon\}.$$

Then the class $\{N_\varepsilon(x) : x \in \omega, \varepsilon > 0\}$ forms a base for a topology T_∞ on ω. Note that convergence in (ω, T_∞) is just uniform convergence. We assume that any subspace c_S of ω is endowed with the relative topology from (ω, T_∞). On the space m and its subspaces this is the usual sup norm topology.

PROPOSITION 4. *If S is a nonnegative summability method and $e \in c_S$, then S is continuous with respect to T_∞.*

Proof. Clearly T_∞ is first countable (in fact it is a metric topology), so we need only show that, if $x^n \to y$ uniformly in c_S, then $S(x^n) \to S(y)$. Given $\varepsilon > 0$, for all large n and for all $i \geq 0$ we have

$$-\varepsilon \leq x_i^n - y_i \leq \varepsilon.$$

Hence $(-\varepsilon)e \leq x^n - y \leq \varepsilon e$ and so, since S is clearly monotonic,

$$-\varepsilon \cdot S(e) = S((-\varepsilon)e) \leq S(x^n) - S(y) \leq S(\varepsilon e) = \varepsilon S(e).$$

Proposition 3 and 4 clearly imply the following.

COROLLARY. *If S is an RSM, then S is continuous with respect to the uniform topology.*

PROPOSITION 5. *If S is an RSM, then, for any $x \in c_S$, we have*

$$\liminf x \leq S(x) \leq \limsup x.$$

Proof. Let $x \in c_S$. If $\liminf x = -\infty$ then, clearly, $\liminf x \leq S(x)$. Suppose that $\liminf x > -\infty$. For each n, let $y_n = \inf_{k \geq n} x_k$. Each y_n is real, $y = (y_n) \leq x$ and $\lim y = \liminf x$. Next consider the eventually constant sequence z^n defined by

$$z_i^n = \begin{cases} y_i & \text{if } i \leq n, \\ y_n & \text{if } i > n. \end{cases}$$

Then $z^n \in c \subseteq c_S$ and $S(z^n) = \lim z^n = y_n$. Since $z^n \leq x$ we obtain $y_n \leq S(x)$ for each n from which it follows that $\liminf x = \lim y \leq S(x)$. The left hand half of the result is thus proved. The other half is

$$S(x) = -S(-x) \leq -\liminf -x = \limsup x.$$

The proof of Proposition 5 shows that, if $m \cdot |c_S|^0 = |c_S|^0$, $c \subseteq c_S$, and $S(x) = r$ whenever $x = (x_1, x_2, \ldots, x_k, r, r, r \ldots)$, then $\liminf x \leq S(x) \leq \limsup x$ for all $x \in c_S$ and, so, S is regular, whence an RSM.

It also follows from Proposition 5 that if S is an RSM, then its domain c_S cannot be too large.

COROLLARY. *If S is an RSM, then $c_S \neq \omega$.*

Proof. For example $x = (1, 2, 3, 4 \ldots\ldots)$ cannot be a member of c_S.

On the other hand, if c_S is, in a certain sense, small, then we can weaken the condition that $|c_S|^0$ be solid and still obtain an RSM as is done in the next proposition.

Proposition 6. *Let S be a summability method such that there exists a bounded sequence not in the domain of S (i.e. $m \not\subseteq c_S$). Then, if S is regular and $m \cdot |c_S|^0 \subseteq c_S$, we have that S is an RSM.*

Proof. After Proposition 2, it is sufficient to let $x \in |c_S|^0$, $|y| \leq |x|$ and show that $y \in |c_S|^0$. By the hypotheses and Proposition 1, we have $|y| \in c_S$.

Suppose $S(|y|) = -r < 0$. Then $|y| + r \geq 0$ and, $S(|y| + r) = 0$. By definition $|y| + r \in |c_S|^0$. Let $z \in m$ be any bounded sequence. Then the sequence

$$\frac{z}{|y|+r} = \left(\frac{z_i}{|y_i|+r}\right)$$

is also a bounded sequence and so

$$z = \frac{z}{|y|+r} \cdot (|y|+r) \in m \cdot |c_S|^0 \subseteq c_S,$$

hence $m \subseteq c_S$ contrary to hypothesis. Therefore $S(|y|) \geq 0$.

If $S(|y|) = r > 0$, then let $u = |x| - |y|$. We have $|u| = u \leq |x|$. As before, $|u| \in c_S$ and $S(|u|) = 0 - r < 0$. This again leads to the contradiction that $m \subseteq c_S$. Hence $S(|y|) = 0$ and $y \in |c_S|^0$.

In the next section we shall look at several examples of RSMs and non RSMs. In particular, in connection with Proposition 6, Example 1 gives a summability method which is regular, $m \cdot |c_S|^0 \subseteq c_S$ but S is not an RSM. However, if S is regular, $m \cdot |c_S|^0 \subseteq c_S$ and S is non-negative, then S is an RSM as the next proposition shows:

Proposition 7. *If S is regular, nonnegative and $m \cdot |c_S|^0 \subseteq c_S$, then S is an RSM.*

Proof. Again, if $x \in |c_S|^0$ and $|y| \leq |x|$, we have $|y| \in c_S$ and $0 \leq S(|y|) \leq S(|x|) = 0$. Hence $y \in |c_S|^0$.

Proposition 7 implies that any generalized limit (nonnegative regular linear functional on m, see [3]) is an RSM.

Replacing nonnegative with continuous in Proposition 7 will not yield a theorem as is shown below by Example 3.

3. Examples

In this section we present examples of regular summabilities which help distinguish the conditions $m|c_S|^0 \subseteq c_S$, $m|c_S| \subseteq |c_S|^0$, $m \not\subseteq c_S$, continuity (under T_∞), nonnegativity, and being an RSM.

The first example shows that S regular, $m \cdot |c_S|^0 \subseteq c_S$, $m \subseteq c_S$ are not sufficient conditions for S being an RSM.

EXAMPLE 1. Write $\omega = c \oplus d$. For any $x \in \omega$ we can write uniquely $x = x^c + x^d$ where $x^c \in c$ and $x^d \in d$.

Define a linear functional S on ω by $S(x) = \lim x^c$. Then $c_S = \omega$ and S is a regular summability method with $m \cdot |c_S|^0 \subseteq c_S$ and $m \subseteq c_S$. By the corollary to Proposition 5, S is not an RSM.

In the next example we show that S being regular and nonnegative (where we even have that c_S is a closed subspace of (ω, T_∞)) does not imply that $m \cdot |c_S|^0 \subseteq c_S$.

EXAMPLE 2. Let $A \subseteq I$ be an infinite set of positive integers such that its complement $I \backslash A$ is also infinite and let $c_S = c \oplus \langle \chi_A \rangle$ where χ_A is the caracteristic sequence of A and $\langle \chi_A \rangle$ denotes the subspace of ω spanned by χ_A. Let S be defined by $S(x+t) = \lim x$, where $x \in c$ and $t \in \langle \chi_A \rangle$. One easily checks that S is nonnegative and regular and that $c \oplus \langle \chi_A \rangle$ is closed. Note that $S(\chi_A) = 0$. Now let $B \subseteq A$ such that B and $A \backslash B$ are infinite. Clearly $\chi_B \in m \cdot |c_S|^0$. Suppose that $\chi_B \in c_S$ and so $\chi_B = x + r\chi_A$ where $x \in c$. Then $x = \chi_B - r\chi_A$ and x has infinitely many terms with value $1-r$ and infinitely many with value $-r$. This is impossible since x is convergent. In this example $|c_S|^0 = c_0 \oplus \langle \chi_A \rangle$ which is a subspace of c_S but not solid, in fact $m \cdot |c_S|^0 \subsetneq c_S$.

By the next example we conclude that a regular summability method S being continuous and satisfying $m \cdot |c_S|^0 \subseteq c_S$ does not imply that S is nonnegative.

EXAMPLE 3. Suppose that f and g are continuous, regular linear functionals on m (e.g. Banach limits). Let h be the summability method on m defined by

$$h(x) = 2f(x_1, x_3, \ldots, x_{2n+1}, \ldots) - g(x_2, x_4, x_6, \ldots, x_{2n}, \ldots).$$

Then h is continuous, regular and $m \cdot |c_S|^0 \subseteq m = c_S$, but h is not nonnegative since $h(0,1,0,1\ldots)=2\cdot 0-1=-1$.

In this example $|c_S|^0$ is not a linear space since $\left(\frac{1}{2},1,\frac{1}{2},1,\ldots\right)$ and $\left(-\frac{1}{2},1,-\frac{1}{2},1,\ldots\right)$ are both in $|c_S|^0$ but their sum is not.

4. Matrix Methods

For any regular matrix A, we define a regular summability f_A on the convergence field c_A by $f_A(x)=\lim(Ax)_n$. If f_A is an RSM we call A an RSM matrix. If A is a nonnegative (i.e. $a_{ij}\geq 0$) regular matrix then A is an RSM matrix. But when a regular matrix A is not nonnegative, necessary and sufficient conditions for f_A to be an RSM are not at all clear. In this section we present examples and propositions concerning regular matrix methods.

Recall that if A is regular, then f_A is a continuous regular summability on (c_A, T_∞).

Example 4. Let us consider the regular matrix A given by

$$A=\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{16}\cdots \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8}\cdots \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4}\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2}\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

If $x=(1,2,1,2,1,2,\ldots)$ and $y=(1,1,1,2,1,1,1,2\ldots)$, then $Ax=0$, $|y|\leq|x|$ and $Ay=\left(\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\ldots\right)$ whence $x\in m\cdot|c_A|^0$ and $y\in m\cdot|c_A|^0$ but $y\notin c_A$. Thus $m\cdot|c_A|^0\not\subset c_A$.

Also, if we take $x=(0,1,0,1,0,1,\ldots)$ then $Ax=(-1,-1,-1,\ldots)$. Therefore f_A is not nonnegative. This example can be generalized as follows.

Proposition 8. *Let A be a regular matrix with the following two properties*: (1) *For each column of A, the members of that column are* *either all nonnegative or all nonpositive and* (2) $\lim_n \sum_{k=1}^{\infty} a_{nk}^{-} = r > 0$ (*where* $a^{+} = \max(a, 0)$ *and* $a^{-} = (-a)^{+}$). *Then* f_A *cannot be nonegative nor can we have* $m \cdot |c_A|^0 \subseteq c_A$.

Proof. Let $x \in \omega$ be defined by

$$x_k = \begin{cases} 0 & \text{if } k\text{-th column of } A \text{ is nonnegative or zero,} \\ 1 & \text{if } k\text{-th column of } A \text{ is nonpositive.} \end{cases}$$

Then

$$\lim_n (Ax)_n = \lim_n \left(\sum_{k=1}^{\infty} a_{nk}^{+} x_k - \sum_{k=1}^{\infty} a_{nk}^{-} x_k \right)$$
$$= -\lim_n \sum_{k=1}^{\infty} a_{nk}^{-} = -r < 0.$$

Therefore $x \in c_A$ and $f_A(x) = -r$. Hence f_A is not nonnegative.

To prove that $m \cdot |c_A|^0 \not\subseteq c_A$ we proceed as follows. By the standard gliding hump technique we can find two sequences of positive integers $K_1 < K_2 < K_3 < \dots$ and $N_1 < N_2 < N_3 < \dots$ such that

$$\sum_{j=1}^{K_i} |a_{N_i j}| < \frac{1}{i} \text{ and } \sum_{j=K_i+1}^{\infty} |a_{N_i j}| < \frac{1}{i}.$$

Define $y_j = r$ if $K_{2i} \leq j < K_{2i+1}$, for some i, and $y_j = 0$ otherwise. If we further define

$$x_j = \begin{cases} 1+r & \text{if the } j\text{th column of } A \text{ is non-positive,} \\ r & \text{otherwise,} \end{cases}$$

then $0 \leq y \leq x$ and, firstly,

$$f_A(x) = r \cdot \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}^{+} - (1+r) \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}^{-} = 0,$$

so that $x \in |c_A|^0$ and, secondly,

$$(Ay)_{N_{2i}} \to r \text{ and } (Ay)_{N_{2i+1}} \to 0 \quad (i \to \infty)$$

so that $y \notin c_A$. This completes the proof.

Both conditions in the above proposition are required: A nonnegative regular matrix is an RSM matrix and satisfies the first condition but not the second. The next example is a matrix which satisfies the second but not the first and turns out also to be an RSM.

EXAMPLE 5. Let A be given by

$$A=\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0\dots & & & \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0\dots & & & \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\dots \\ \vdots & & & \vdots & & & \vdots & & & \\ \vdots & & & \vdots & & & \vdots & & & \\ \vdots & & & \vdots & & & & & & \end{bmatrix}$$

If d_n is defined to be $(x_1+x_2+\dots+x_n)/n$, then, for any x, the nth term of Ax is clearly $3d_{3n}-2d_n$. We show that, for $x\geq 0$, Ax converges to zero if and only if d_n converges to zero. This shows that $|c_A|^0=|\sigma_1|^0$ (the strongly cesaro summable (to zero) sequences–see [1]) and thus $m\cdot|c_A|^0=|c_A|^0$. Obviously, if $d_n\to 0$, then $3d_{3n}-2d_n\to 0$.

Now let $x\geq 0$ and suppose $3d_{3n}-2d_n\to 0$. By the nonnegativity of x we obtain, for all n, that

$$d_{n+1}\geq\frac{n}{n+1}d_n. \tag{*}$$

First suppose d_n is unbounded. Choose an N such that $N\geq 2$ and, if $d_n>N$, then $(n-2)/n>5/6$ and $3d_{3j}-2d_j<1$ for all $j\geq n/3$. Let n be the first index such that $d_n>N$. If $n=3p$, then

$$1>3d_{3p}-2d_p>3d_p-2d_p=d_p$$

whence

$$d_n=d_{3p}<\frac{1+2d_p}{3}<1<N.$$

If $n=3p-1$, then, using (*),

$$\begin{aligned}1>3d_{3p}-2d_p&\geq 3\cdot\frac{3p-1}{3p}d_{3p-1}-2d_p\\ &>\left(3\cdot\frac{3p-1}{3p}-2\right)d_p>\left(\frac{5}{2}-2\right)d_p=\frac{d_p}{2},\end{aligned}$$

and

$$d_n\leq\frac{3p}{3p-1}d_{3p}<\frac{6}{5}\left(\frac{1+2d_p}{3}\right)<2\leq N.$$

A similar contradiction is obtained if $n=3p-2$. It follows that d_n is a bounded sequence.

Now let $u=\limsup_n d_n$. For any $\varepsilon>0$, we have $d_n\geq u-(\varepsilon/2)$ for infinitely many n. From this and (*) we conclude that $d_{3n}\geq u-\varepsilon$ for infinitely many n. Since $d_n \leq u+\varepsilon$ for all large n, we have $3d_{3n}-2d_n \geq 3(u-\varepsilon)-2(u+\varepsilon)=u-5\varepsilon$ for infinitely many n. This implies that $u=0$ and hence that d_n converges to zero.

Examples can also be found of matrices A satisfying exactly one or none of the conditions of Proposition 8 such that A is not an RSM matrx. The following is one such.

EXAMPLE 6. Let

$$A=\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0\cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0\cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3}\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Let $x\in\omega$ be defined by $x=(0,0,0,1,0,2,0,3,0,4,0,\ldots)$. Then $(Ax)_n=-1$ for all n, so that $x\in c_A$ and $f_A(x)=-1$. Hence f_A is not nonnegative and thus f_A is not an RSM. Furthermore, if we let

$$x=(1,1,1,2,1,3,1,4,1,5,1,6,1,7,\ldots)$$
$$y=(1,1,1,1,1,3,1,1,1,5,1,1,1,7,\ldots)$$

then $(Ax)_n=0$ for all n so that $x\in|c_A|^0$. But $y\notin c_A$ since $(Ay)_n=1$ if n is odd and $=0$ if n is even. Thus $m\cdot|c_A|^0\not\subseteq c_A$.

The previous example shows that a regular matrix which is "essentially nonnegative" may still not be an RSM. Essentially nonnegative means that $\lim_n \sum_k a_{nk}^-=0$. However, as is often the case with matrix summabilities, if we restrict the domain to the bounded convergence field, i.e., $c_A\cap m$, then f_A becomes an RSM. Firstly, if A is an essentially nonnegative regular matrix and $A^+=(a_{nk}^+)$ (which is nonnegative and regular), then $c_A\cap m=c_{A^+}\cap m$ and $f_A(x)=f_{A^+}(x)$ for all $x\in c_A\cap m$. We omit the straightforward proof.

Furthermore, we have

PROPOSITION 9. *Let S be an RSM on c_S and let T be the restriction of S to the domain $c_T=c_S\cap m$. Then $|c_T|^0=|c_S|^0\cap m$ and T is an RSM.*

Proof. That $|c_S|^0\cap m\subseteq|c_T|^0$ is clear. If $x\in|c_T|^0$ then $|x|\in c_T$ and $T(|x|)=0$. Hence x is bounded and $S(|x|)=0$ so that $x\in|c_S|^0\cap$

m. Now let $x \in |c_T|^0$ and $b \in m$. Then $bx \in m$ and $bx \in |c_S|^0$ so that $m \cdot |c_T|^0 \subseteq |c_T|^0$.

From proposition 9 and the remarks preceding it we obtain our concluding result.

PROPOSITION 10. *If A is an essentially nonnegative matrix and S is the restriction of f_A to the domain $c_A \cap m$, then S is an RSM.*

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