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SOME CYCLIC GROUP ACTIONS ON HOMOTOPY SPHERES

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In [4] Orlik defined a free cyclic group action on a homotopy sphere constructed as a Brieskorn manifold and proved the following theorem:

THEOREM. *Every odd-dimensional homotopy sphere that bounds a parallelizable manifold admits a free $\boldsymbol{Z}_p$-action for each prime p.*

On the other hand, it was shown ([3]) that there exists a free $\boldsymbol{Z}_p$-action on a $2n$-1 dimensional homotopy sphere so that its orbit space is stably parallelizable if and only if $n \leq p$. Naturally, one can ask:

QUESTION. Does each odd dimensional homotopy sphere $\sum^{2n-1}$, $n>2$ that bounds a parallelizable manifold admit a free $\boldsymbol{Z}_p$-action so that its orbit space is stably parallelizable whenever $n \leq p$?

As a partial answer of the question and as a generalization of Orlik's theorem for $4n-3$, $n \geq 2$, dimensional homotopy spheres, we will show:

MAIN THEOREM. *For $n \geq 3$ odd, $p \geq n$, any homotopy sphere of dimension $2n$-1 that bounds a parallelizable manifold admits a free $\boldsymbol{Z}_p$-action so that its orbit space is stably parallelizable.*

To prove it, we need a generalized Brieskorn manifold. Let

$$f_i(z_1, \cdots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}}, \quad i=1, \ldots, m$$

be polynomials having only one critical point at the origin, where a_{ij} are integers greater than 1, a_{ij} are real numbers, and $n>2$. Set

$$V_i = \{z \in \boldsymbol{C}^{n+m} : f_i(z) = 0\},$$
$$V(a) = V_1 \cap V_2 \cap \cdots \cap V_m,$$

and

$$\sum(a) = V(a) \cap S^{2(n+m)-1}.$$

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Then, by Whitney's theorem, $V_i-\{0\}$ is a smooth manifold of dimension $2(n+m)-3$. This $\Sigma(a)$ is called a generalized Brieskorn manifold, and a generalized Brieskorn sphere if it is a homotopy sphere. The gradient of $f(z_1, \ldots, z_{n+m})$ is defined by

$$\text{grad } f(z) = (\overline{\partial f/\partial z_1}, \ldots, \overline{\partial f/\partial z_{n+m}})_z$$

where $\overline{\partial f/\partial z_i}$ is the complex conjugate of $\partial f/\partial z_i$.

For our purpose, we assume that grad $f_1, \ldots,$ grad f_m are linearly independent at each point $V(a)-\{0\}$, and $\Sigma(a)$ is a homotopy sphere (of dimension $2n-1$).

Let a_{ij} be independent of i, say, $a_{ij}=a_j$ for all i, j, and assume that the real matrix (α_{ij}) has no zero subdeterminant. Construct a graph $G(a)$ as follows: $G(a)$ has $n+m$ vertices $a_1, a_2, \ldots, a_{n+m}$ and edge $\overline{a_i a_j}$ whenever the greatest common divisor (a_i, a_j) is greater than one. Let $\#_a$ be the number of the connected components K of $G(a)$ such that K has an odd number of vertices and for any two different vertices a_i, a_j of K, $(a_i, a_j)=2$. Then, (by Hamm's theorem [2]) $\#_a > m$ implies that $\Sigma(a)$ is a topological sphere.

Let p be an odd prime, and let $\Sigma(a)$ be such a topological sphere with $a_{ij} \neq 0 \pmod p$. Define a free $\boldsymbol{Z}_p$-action on $\Sigma(a)$ as follows: Choose natural numbers b_j so that $a_{ij}b_j=1 \pmod p$ for all i, j. Define an action of $\boldsymbol{Z}_p$ on $\Sigma(a)$ by

$$\zeta(z_1, \ldots, z_{n+m}) = (\zeta^{b_1}z_1, \ldots, \zeta^{b_{n+m}}z_{n+m}),$$

where ζ is a primitive p-th root of unity considered as the preferred generator of $\boldsymbol{Z}_p$. Clearly, $\Sigma(a)$ is invariant under this action. The orbit space $\Sigma(a)/\boldsymbol{Z}_p$ is a homotopy lens space and will be denoted by $L(p\,;a\,;b)$.

The stable parallelizability problem of these homotopy lens spaces was answered as follows:

THEOREM ([3]). *A $2n-1$ dimensional homotopy lens space $L(p\,;a\,;b)$ is stably parallelizable if and only if*

(1) $n \leq p$, *and*

(2) $b_1^{2j}+b^{2j}\cdots+b_{n+m}^{2j}=m \pmod p$ *for* $j=1, 2, \ldots, \left[\frac{1}{2}(n-1)\right]$.

It is well-known that the group bP_{4k-2} of $4k-3$ dimensional homotopy spheres which bound a parallelizable manifold is either zero or cyclic of order two, i. e., the standard sphere and the Kervaire sphere. Also bP_{4k-2}, $k=\text{odd}\geq 3$, is the cyclic group of order two, equivalently, the Kervaire sphere is exotic.

In the given cases, i. e.,

$$f_i(z_1, \dots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}},$$
$$V_i = f_i^{-1}(0), \quad i=1, 2, \dots, m \text{ and}$$
$$\Sigma(a) = V_1 \cap V_2 \cap \cdots \cap V_m \cap S^{2(n+m)-1},$$

the following classification of the diffeomorphism classes of homotopy spheres is useful.

THEOREM ([2]). *Assume that a_{ij} is independent of i, say $a_{ij}=a_j$ for all i, j and let $G(a)$ be the graph defined as before. If $\Sigma(a)$ is a topological sphere of dimension $2n-1$, $n\geq 3$ odd, then*

(1) *if the graph $G(a)$ has exactly $m+1$ components and the number of one point components a_j with $a_j=\pm 3 \pmod 8$ is odd, then $\Sigma(a)$ is the Kervaire sphere,*

(2) *in all other cases, $\Sigma(a)$ is the standard sphere.*

As a necessary condition for a lens space $L(p\,;a\,;b)$ to be stably parallelizable, there must be a solution to the system of equations

$$b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod p, \quad j=1, 2, \dots, \left[\frac{1}{2}(n-1)\right],$$

over the field $\boldsymbol{Z}_p$, and the existence of their common solution comes from the next lemma.

LEMMA. *Let $p\geq n\geq 3$ and let $q=\left[\frac{1}{2}(n-1)\right]$. Then there exists at least one common solution $(x_1, x_2, \dots, x_{n-m})$ with each $x_i\neq 0$ in $\boldsymbol{Z}_p$ to*

$$(\mathrm{I}) \qquad \begin{cases} x_1^2 + \cdots + x_n^2 + \cdots + x_{n+m}^2 = m \pmod p, \\ x_1^4 + \cdots + x_n^4 + \cdots + x_{n+m}^4 = m \pmod p, \\ \quad \cdots \\ x_1^{2q} + \cdots + x_n^{2q} + \cdots + x_{n+m}^{2q} = m \pmod p, \end{cases}$$

for some $m\geq 1$.

Proof. If $n=p$, then $x_1=x_2=\cdots=x_{n+m}=1$ is a solution for any $m\geq 1$. Let $n<p$, and let $c_1=1$, $c_2, \ldots, c_{(p-1)/2}$ be the quadratic residue in $\boldsymbol{Z}_p$. Since each equation in (I) is homogeneous of even order, the system (I) can be reduced to the following system:

$$(\mathrm{II})\quad \begin{cases} y_1+y_2+\cdots+y_{(p-1)/2}=n+m, \\ y_1+c_2y_2+\cdots+c_{(p-1)/2}y_{(p-1)/2}=m \pmod{p}, \\ y_1+c_2^2y_2+\cdots+c^2_{(p-1)/2}y_{(p-1)/2}=m \pmod{p}, \\ \cdots \\ y_1+c_2^qy_2+\cdots+c^q_{(p-1)/2}y_{(p-1)/2}=m \pmod{p}. \end{cases}$$

Indeed, each y_j in the system (II) represents the number of x_i's with $x_i^2=c_j \pmod{p}$ in the system (I), so that the first equation in (II) must be added. Now it is enough to show the existence of a solution to the system (II). To do this, it can be considered as a system of equations in the field $\boldsymbol{Z}_p$, hence it is needed to change the first equation in (II) to the equation in $\boldsymbol{Z}_p$, i.e.,

$$y_1+y_2+\cdots+y_{(p-1)/2}=n+m \pmod{p}.$$

If the system (II) with this equation instead of its first equation, say (II)$'$, has a solution, then (II) has a solution $(y_1, y_2, \ldots, y_{(p-1)/2})$ with

$$y_1+y_2+\cdots+y_{(p-1)/2}=n+m+kp$$

for some k. By taking a sufficient large number y_1, i.e., adding more variables x_j with $x_j^2=c_1=1 \pmod{p}$ in the system (I), we can assume that $k\geq 0$, and then (I) will have a solution with $m+kp$, $k\geq 0$ instead of m. First, consider the case of $q+1=\frac{1}{2}(p-1)$, then the coefficient matrix of the system (II)$'$ has a nonzero Vandermonde determinant. So it has a solution. The remaining case of $q+1<\frac{1}{2}(p-1)$ can be proved by the same method by taking

$$y_{q+2}=y_{q+3}=\cdots=y_{(p-1)/2}=0.$$

To prove the main theorem, let $3\leq n\leq p$, and let $b_1, b_2, \ldots, b_{n+m}$, $(m\geq 1)$ be nonzeros in $\boldsymbol{Z}_p$ such that

$$b_1^{2j}+b_2^{2j}\cdots+b_{n+m}^{2j}=m \pmod{p}, \quad j=1,2,\ldots,\left[\frac{1}{2}(n-1)\right].$$

By Dirichlet's theorem, there exist distinct prime numbers $a_1, a_2, \ldots, a_{n+m}$ such that $a_j b_j=1 \pmod{p}$ for $j=1,2,\ldots, n+m$. Now, one can construct m polynomials

$$f_i(z_1, \ldots, z_{n+m})=\sum_{j=1}^{n+m}\alpha_{ij}z_j^{a_{ij}}, \quad i=1,2,\ldots,m$$

with $a_{ij}=a_j$ for all i and real numbers α_{ij} chosen so that the matrix (α_{ij}) has no zero subdeterminent. Then, by Hamm's theorem,

$$\Sigma(a)=V_1\cap V_2\cap\cdots\cap V_m\cap S^{2(n+m)-1},$$

where $V_i=f_i^{-1}(0)$, is a $2n-1$ dimensional homotopy sphere, and clearly grad f_1, grad $f_2, \ldots$, grad f_m are linearly independent. Hence we have the following result: "For $n\geq 3$ odd, $p\geq n$, every $2n-1$ dimensional standard sphere admits a free $\boldsymbol{Z}_p$-action so that its orbit space is stably parallelizable". Next, we will see that it is also true for the exotic spheres. Note that we can assume $b_1=b_2=\cdots=b_n=1$ by taking more polynomials if necessary. Consider a set

$$S=\left\{\frac{1}{2}(2k+1)p+\frac{1}{2} : k\in \boldsymbol{N}\right\}=\left\{\frac{1}{2}(p+1)+kp : k\in \boldsymbol{N}\right\}.$$

This set S contains infinitely many prime numbers by Dirichlet's theorem. Choose distinct primes $q_1, q_2, \ldots, q_n$ in S so that $q_i>a_{n+j}$ for all i, j. Now, take $a_i'=2q_i$, $i=1,2,\ldots,n$ so that $(a_i', a_j)=2$ for $1\leq i\neq j\leq n$, and $a_i'b_i=a_i'=2q_i=2\left(\frac{1}{2}(p+1)+k_ip\right)=1 \pmod{p}$ for $1\leq i\leq n$. Take $a'_{n+j}=a_{n+j}$ for $j=1,2,\ldots,m$. Then, the graph $G(a')$ has exactly $m+1$ components: $\{a_1', a_2', \ldots, a_n'\}$ and single components $a'_{n+1}, \ldots, a'_{n+m}$. If the number of single components a'_{n+j} with $a'_{n+j}=\pm 3 \pmod{8}$ is odd, then $\Sigma(a')$ is the Kervaire sphere. If not, we can take a_{n+m+1}, b_{n+m+1} so that $a_{n+m+1}=3 \pmod{8}$, and a_{n+m+1} is a prime number greater than all other a_i's and $a_{n+m+1}=1 \pmod{p}$. Indeed, by Chinese Remainder Theorem, $x=1 \pmod{p}$ and $x=3 \pmod{8}$ has a solution, say x_0, then $x_0+8p\boldsymbol{N}$ contains infinitely many prime numbers, because $(x_0, 8p)=1$. Now one can choose a sufficient large number among these primes as a_{n+m+1}. Clearly, by adding this number to (a_i), one has a Kervaire sphere. The remaining process is exactly the same as before. This proves the main theorem.

References

1. Brieskorn E., *Beiespiele zur Differential-Topologie von Singularitäten*, Inventiones Math. **2**(1966), 1-14.
2. Hamm H. A., *Exotische Sphären als Umgebungsränder in speziellen Komplexen Räumen*, Math. Ann. **197**(1972), 44-56.
3. Kwak J. H., *The stable parallelizability of a smooth homotopy lens space*, J. of Pure and Applied Algebra **50**(1988), 155-169.
4. Orlik P., *Smooth homotopy lens spaces*, Michigan Math. J. **16**(1969), 245-255.

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