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# COMPRESSIBILITY AND ANNIHILATOR CONDITIONS

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Throughout this paper $R$ denotes a ring with identity. Motivated by S. K. Berberian's question in [1], we are concerned with the cardinality of idempotents of regular Baer rings. Actually we show that regular Baer rings with countably many idempotents are semisimple Artinian, thereby we can generalize Rangaswamy's theorem [11]. As a byproduct of this result we can give a partial answer to S. K. Berberian's conjecture in [1], that is, every regular Baer ring with countably many idempotents is compressible by adapting L. Jeremy's result [5].

Moreover we study right nonsingular rings satisfying certain annihilator conditions. Several characteristic properties of a semisimple Artinian ring are obtained in terms of annihilators. Consequently we have a slight generalization of some results in [13], [12] and [9].

## 1. Compressibility of regular Baer rings

For a subset $S$ of a ring $R$ the set $r(S)=\{x\in R\mid Sx=0\}$ is called the right annihilator of $S$ in $R$. Of course $r(S)$ is a right ideal of $R$. The left annihilator $l(S)$ of $S$ is defined analogously. The right (resp. left) ideal of the form $r(S)$ (resp. $l(S)$) for some subset $S$ of $R$ is called the right (resp. left) annihilator ideal of $R$.

Definition 1.1. A ring $R$ is called Baer if every right annihilator is of the form $eR$, $e$ an idempotent.

Definition 1.2. A ring $R$ is called (von Neumann) regular if for every $x$ in $R$ there exists $y$ in $R$ such that $x=xyx$.

Since the following statements are quite strightforword, its proof

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will be omitted.

THEOREM 1.3. *Let $R$ be a Baer ring.*
*Then the following are equivalent.*

(1) *$R$ is regular.*

(2) *Every finitely generated right ideal is a right annihilator.*

(3) *Every principal right ideal is a right annihilator.*

COROLLARY 1.4. *A ring $R$ is left self-injective Baer if and only if it is left self-injective regular.*

By definitions we know that Baer rings and regular rings are greatly influenced by the structure of idempotents. In the following one of our main results is that a regular Baer ring with countably many idempotents is indeed semisimple Artinian ring. But of course we have to note that there is a semisimple Artinian ring with uncountably many idempotents, for example the $2\times 2$ matrix ring over the reals.

THEOREM 1.5. *A regular Baer ring with countably many idempotents is semisimple Artinian.*

*Proof.* If $R$ satisfies the ACC on finitely generated right ideals then we are done. Suppose $R$ does not satisfy the ACC on finitely generated right ideals. We thus have a countably generated right ideal $A$ which can not be finitely generated. Write $A$ as the union of an ascending chain of finitely generated right ideals,

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots\cdots$$

First choose any idempotent $e_1$ in $R$ such that $A_1 = e_1R$. Since $e_1R \subseteq A_2$, we see that $A_2 = e_1R \oplus (1-e_1)A_2$. In addition, $R = A_2 \oplus B$ for some right ideal $B$; hence, there exists an idempotent $e_2$ in $R$ such that $e_2R = (1-e_1)A_2$ and $(1-e_2)R = e_1R \oplus B$. Clearly, $e_1e_2 = e_2e_1 = 0$ and $e_1R \oplus e_2R = A_2$. Since $(e_1+e_2)R = A_2$, we may use the same argument to find an idempotent $e_3$ in $R$ such that $e_3$ is orthogonal to $e_1+e_2$ and $(e_1+e_2)R \oplus e_3R = A_3$. Obviously, $e_1, e_2, e_3$ are orthogonal and $e_1R \oplus e_2R \oplus e_3R = A_3$. Continuing in this fashion, we eventually obtain orthogonal idempotents $e_1, e_2, e_3, \ldots$ in $R$ such that $e_1R \oplus \ldots \oplus e_nR = A_n$ for all $n$. Clearly $\oplus e_nR = \cup A_n = A$. Since $A$ is not finitely generated, infinitely many $e_n \neq 0$, giving us an infinite sequence of nonzero orthogonal idempotents. Write this sequence of nonzero orthogonal idempotents as $\{f_n \mid n \in N\}$.

Since $R$ is Baer, for any nonempty subset $S$ of $N$, we can get an idempotent $g_s$ in $R$ such that $r(\{f_i | i \in S\}) = (1-g_s)R$. By definition, $f_i(1-g_s)=0$ for all $i$ in $S$, i.e., $f_i = f_i g_s$, for all $i$ in $S$. Now for every $k \notin S$, $i \in S$, $f_i f_k = 0$. Thus $f_k \in (1-g_s)R$; hence $g_s f_k = 0$ for every $k \notin S$. Since $f_k g_s f_k = 0$, $f_k g_s \neq f_k$ for every $k \notin S$. Hence if $S$ and $T$ are distinct subsets of $N$, then $g_s \neq g_t$. Thus $R$ has a uncountable set of idempotents. It is a contradiction to the supposition.

As an immediate corollary of theorem 1.5., we get following.

Corollary 1.6. (11, K. M. Rangaswamy). *Any countable regular Baer ring is semisimple Artinian.*

As a byproduct from the process of the proof of theorem 1.5. the following corollary follows immediately.

Corollary 1.7. *If Baer ring $R$ has a nonzero infinite sequence of orthogonal idempotents, then $R$ has an uncountable set of idempotents.*

A ring $R$ (with identity) is compressible if for each idempotent $e$ of $R$, $Z(eRe) = eZ(R)$, where $Z(R)$ denotes the center of the ring $R$.

When every idempotent is central then obviously $R$ is compressible. So in some sense the compressibility is a kind of measure for the centrality of idempotents. The concept of compressible rings was initially introduced by S. K. Berberian in [1] and several classes of rings related to operator algebras were shown to be compressible. After his results in [1] S. K. Berberian raised following conjecture "Is every regular Baer ring compressible?"

With our observation on the idempotent structure of regular Baer rings we give a partial affirmative answer to the conjecture by adapting following L. Jeremy's result in [5].

Proposition 1.8. (5, L. Jeremy). *Any regular (right) self-injective ring is compressible.*

Now we can give a partially affirmative answer for S. K. Berberian conjecture.

THEOREM 1. 9. *A regular Baer ring with countably many idempotents is compressible.*

## 2. Annihilator conditions in right nonsingular rings

We begin with a definition due to C. R. Yohe [12].

DEFINITION 2. 1. A ring $R$ is a left elemental annihilator ring (l. e. a. r.) if, whenever $L$ is a left ideal of $R$, there exists an element $a \in R$ such that $L = l(a)$. A right elemental annihilator ring (r. e. a. r.) is defined analogously.

Note that any semisimple Artinian ring is a left and right elemental annihilator ring. Indeed, if $L$ is a left ideal of $R$, $L = Re$ where $e$ is an idempotent and so $L$ is exactly the left annihilator of the element $1-e$.

It is well known that a semiprime ring with ACC on right annihilators is a right nonsingular ring.

THEOREM 2. 2. *Let R be a semiprime with ACC on right annihilators. Then the following are equivalent.*

(1) *$R$ is semisimple Artinian.*

(2) *Every principal left ideal of $R$ is a left annihilator.*

(3) *Every finitely generated left ideal of $R$ is a left annihilator.*

(4) *Every left ideal of $R$ is a left annihilator.*

(5) *$R$ is l. e. a. r.*

*Proof*. (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) is clear. And (1) $\Longrightarrow$ (5) is also obvious. Thus it is enough that if we show that (2) implies (1). First observe that ACC on right annihilators is equivalent to DCC on left annihilators. Thus $R$ is right perfect ring satisfying ACC on right annihilators. By [3, Proposition 1], $R$ is semiprimary. Hence $R/J(R)$ is semisimple Artinian and $J(R)$ is nilpotent. But since $R$ is semiprime, $R$ is semisimple Artinian.

COROLLARY 2. 3. (12, C. R. Yohe). *Let R be a semiprime ring with ACC on right annihilators. Then the following are equivalent.*

(1) *$R$ is l. e. a. r.*

(2) *Every left ideal of R is the annihilator of some subset.*

Recall that the right singular ideal of $R$ is $Z_r(R)=\{a\in R\,|\,r(a)$ is an essential right ideal of $R\}$. The next is a slight generalization of some results in [13, Theorem 1], [12, Theorem 2] and [9, Theorem 5.2.]

THEOREM 2.4. *If $R$ is a right nonsingular ring, then the following conditions are equivalent.*

(1) *$R$ is semisimple Artinian.*
(2) *$R$ is r.e.a.r.*
(3) *Every right ideal of $R$ is a right annihilator.*
(4) *Every essential right ideal of $R$ is a annihilator.*
(5) *$R$ has no proper dense right ideal.*
(6) *Every maximal right ideal of $R$ is a right annihilatcr.*
(7) *$l(M)\neq 0$, for every maximal right ideal $M$ of $R$.*

*Proof*. Obviously, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). And (3) $\Longrightarrow$ (6) $\Longrightarrow$ (7) is also clear.

(4) $\Longrightarrow$ (5). Let $I$ be an essential right ideal of $R$.
Then $I=r(S)$ for some subset $S$ of $R$. Since $Z_r(R)=0$, $S=\{0\}$; thus $I=r(0)=R$. Therefore $R$ has no proper dense right ideal, since every dense ideal is essential.

(5) $\Longrightarrow$ (1). Let $I$ be any right ideal of $R$.
Then we have a right ideal $K$ of $R$ such that $I\oplus K$ is essential. Since $Z_r(R)=0$, $I\oplus K$ is a dense right ideal. Thus, $I\oplus K=R$; hence we are done.

(7) $\Longrightarrow$ (1). Let $M$ be any maximal right ideal of $R$.
For any nonzero element $a\in l(M), r(a)\supseteq M$. Since $M$ is maximal, $r(a)=M$. On the other hand $r(a)=M$ can not be essential in $R$. Therefore there exists a nonzero right ideal $K$ such that $M\cap K=0$. Hence $R=M\oplus K$. By [8, Lemma], $R$ is semisimple Artinian.

COROLLARY 2.5. *If $R$ is right nonsingular and a right Kasch ring, then $R$ is semisimple Artinian.*

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