J. Korean Math. Soc. 25(1988), No. 2, pp. 289-301

MAPPING THEOREMS FOR NONLINEAR OPERATORS

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1. Introduction

Let X and Y be Banach spaces and T a nonlinear operator from X into Y. In this paper we consider the global implications of certain local conditions on T and, in particular, derive general conditions under which T is surjective. Our motivation follows from the well-known fact that if T is a local homeomorphism which is a local expansion, in the sense that for a continuous nonincreasing function $c:[0,\infty)\to[0,\infty)$ with $\int_0^\infty c(t)dt=\infty$, each point x in X has a neighborhood U such that

$$c(\max\{\|u\|,\|v\|\})\|u-v\|\leqq\|Tu-Tv\|$$

for each u, v in U, then T is a surjective homeomorphism [5, p. 62].

Kirk and Schöneberg [11], and Ray and Walker [14] proved that the surjectivity of the mapping T can be obtained within the class of mappings whose graphs are closed subsets of $X\times Y$. Also Torrejon [17] obtained the same result without assuming that c is nonincreasing. Moreover, Bae and Yie [3] proved a stronger result by giving the precise range of the operator T, that is, they proved that $T(B(0;K))$ contains $B(T(0);\int_0^K c(t)dt)$ by removing $\int_0^\infty c(t)dt=\infty$, where $B(x;r)$ denotes the open ball of radius r about x. Also they gave the similar result for Gateaux differentiable operators in the same paper [3]. Bae and Sung [2] applied this idea to sums of two nonlinear operators to obtain similar results. However these results are still restricted in the sense that the function c relative to the local expansion constant is homogeneous on the sphere. What happened if we replace the function c by a function whose domain is the whole space X instead

Received March 25, 1988.
Supported by a grant from the Ministry of Education in 1987-88.

of $[0, \infty)$? The main purpose of this paper is to solve this problem. In fact, in [1], the author gave a partial answer restricted to the range of local expansions.

The above results can be thought of as showing that the problem $Tx=y$ is solvable for a suitable y. Concering solvability of the problem we want to calculate the precise range of the operator T. For this purpose, we shall prove the following theorem as a main result as well as a main tool.

THEOREM A. *Let X and Y be complete metric spaces with both metric convexities, D an open subset of X and let $x_0 \in D$. Let $T: D \rightarrow Y$ be a mapping having closed graph. Suppose that $m: D \rightarrow (0, \infty)$ is a continuous function satisfying the condition:*

(1) *there exists an $N>0$ such that if $h: [0, \tau) \rightarrow D$, $0 \leqq \tau \leqq \infty$, is a locally rectifiable curve with $h(0) = x_0$ and*

$$\int_0^\tau m(h(t))\,ds_h(t) < N,$$

then h is rectifiable and $lim_{t \rightarrow \tau^-} h(t) \in D$.

Further suppose that

(2) *for every $x \in D$ and for any sufficiently small $\varepsilon > 0$ with $B(x; \varepsilon) \subset D$ and $k = inf\{m(y) \mid y \in B(x; \varepsilon)\} > 0$, $T(B(x; \varepsilon))$ contains the ball $B(Tx; \varepsilon k)$.*

Then $T(D)$ contains the ball $B(Tx_0; N)$.

This paper is organized as follows. In section 2, we give some definitions and review some properties of rectifiable curves. Section 3 is devoted to the proof of Theorem A. In the remainder sections, we explore wide applications of the above theorem. Especially, we deal with locally expansive operators in section 4 and differentiable operators in section 5.

2. Definitions, Rectifiable Curves

Let X be a metric space. A continuous curve $h: [0, \tau] \rightarrow X$, $0 < \tau < \infty$, is said to be *rectifiable* if there is a constant $M > 0$ such that for any subdivision of $[0, \tau]$ of the form

$$0 = t_0 < t_1 < \ldots < t_n = \tau$$

we have

$$\sum_{i=1}^{n} d(h(t_i), h(t_{i+1})) < M.$$

The least such constant M is said to be the *length* of the curve h. Also a continuous half-open curve $h : [0, \tau) \to X$, $0 < \tau < \infty$, is *rectifiable* if there exists a constant $M > 0$ such that for any sequence of points

$$0 = t_0 < t_1 < \ldots < t_n < \tau$$

we have

$$\sum_{i=1}^{n} d(h(t_i), h(t_{i-1})) < M.$$

The *length* of the curve h is defined by the least such constant M.

A continuous half-open curve $h : [0, \tau) \to X$, $0 < \tau < \infty$, is said to be *locally rectifiable* if for each t in $[0, \tau)$, the restriction curve $h|_{[0, t]}$ is rectifiable. Note that this definition is meaningful when $\tau = \infty$, and any rectifiable curve is locally rectifiable but the converse does not hold. Also in this case we can define the *length function* of the curve h, which is denoted by s_h, that is, $s_h(t)$ is the length of the restriction curve $h|_{[0, t]}$, $0 < t < \tau$. Note that s_h is a continuous nondecreasing real valued function with the same domain of h and $s_h(0) = 0$.

A continuous curve $h : [0, \tau)$ (or $[0, \tau]) \to X$ is said to be *parametrized by arc length* if for any t, $0 \leqq t < \tau$ ($0 \leqq t \leqq \tau$, resp.), we have that $s_h(t) = t$.

Remark 2.1. Note that if X is complete and $h : [0, \tau) \to X$, $0 \leqq \tau \leqq \infty$, is rectifiable, then $\lim_{t \to \tau^-} h(t)$ always exists, and h can be continuously extended to $[0, \tau]$ with the same length when $\tau < \infty$.

Remark 2.2. Let $m : X \to \boldsymbol{R}$ be a continuous function and $h : [0, \tau) \to X, 0 \leqq \tau \leqq \infty$, a locally rectifiable curve. Then we can define the Riemann–Stieltjes integral $\int_0^{\tau} m(h(t)) ds_h(t)$. Moreover, if $t : [0, \delta) \to [0, \tau)$ is a continuous nondecreasing function which is onto, then $g : [0, \delta) \to X$ with $g = h \circ t$ is also locally rectifiable and we have

$$\int_0^{\tau} m(h(t)) ds_h(t) = \int_0^{\delta} m(g(u)) ds_g(u).$$

A metric space X is said to be *rectifiably pathwise conneted* if every pair of points of X can be joined by a continuous rectifiable curve. The definition of a *rectifiably simply connected space* can be given by a similar way (see [5]). Note that every Banach space is rectifiably pathwise connected and every neighborhood of each point contains a subneighborhood which is rectifiably pathwise connected and rectifiably simply connected.

Let X, Y be metric spaces and D a subset of X. Let $T: D \to Y$ be an operator. We say that T is *locally m-expansive* where $m: D \to (0, \infty)$ is a continuous function, if each point x in D has an open neighborhood U such that

(3) $$\min\{m(u), m(v)\} d(u, v) \leq d(Tu, Tv), \quad u, v \in U.$$

And we say that T has *closed graph* if its graph is closed in $D \times Y$, that is, for any sequence $\{x_n\{ \subset D$ with $x_n \to x \in D$ *and* $Tx_n \to y$, it follows that $Tx = y$. Also we say that T has *closed graph in* $X \times Y$ if its graph is closed in $X \times Y$, that is, for any sequence $\{x_n\}$ in D with $x_n \to x$ and $Tx_n \to y$, it follows that $x \in D$ and $Tx = y$.

REMARK 2.3. Note that if T has closed graph in $X \times Y$ then T has closed graph (in $D \times Y$), and if D is closed in X then the converse holds. The property of closed graph is weaker than that of continuity, since every continuous function $T: D \to Y$ has closed graph. However, even if $T: D \to Y$ is continuous, its graph need not be closed in $X \times Y$. But the following lemma can be easily verified.

LEMMA 2.4. *Let T be a mapping from a subset D of a metric space (X, d) into a metric space (Y, d). Define a new metric ρ on D by $\rho(x, y) = max\{d(x, y), d(Tx, Ty)\}$. Then the followings are true.*

(*a*) *$T: (D, \rho) \to Y$ is continuous if τ has closed graph.*

(*b*) *If (D, d) and (Y, d) are complete and T has closed graph, then (D, ρ) is complete.*

(*c*) *If (X, d) and (Y, d) are complete and T has closed graph in $X \times Y$, then (D, ρ) is complete.*

Following Menger [12], a metric space X is said to be *metrically convex* if for each x, y in X with $x \neq y$, there exists a z in X, distinct from x and y, such that $d(x, y) = d(x, z) + d(z, y)$. It is well-known that if X is a complete metric space with metric convexity, then for any x, y in X, there exists an isometry $\sigma: [0, d(x, y)] \to X$ such that $\sigma(0) = x$ and $\sigma(d(x, y)) = y$. We denote by $B(x: r)$ the open ball of radius r about x, and $\bar{B}(x:r)$ its closure. Also conveniently we set $B(x:\infty)=X$.

3. Proof of Theorem A

To prove the basic mapping theorem, Theorem A, we adopt the idea of maximal element technique, in particular, the Brézis–Browder ordering principle [4].

LEMMA 3.1. *Let $\lhd$ be a reflexive relation on a nonempty set M and ϕ : $M \to R$ a function bounded from below, which satisfies the following two conditions*:

(4) *if $x \lhd y$ and $x \neq y$, then $\phi(x) > \phi(y)$ and*

(5) *for any sequence $\{x_n\}$ with $x_n \lhd x_{n+1}$, $n=1, 2, \ldots$, there is a z in M such that for any positive integer k there is an integer $m>k$ with $x_m \lhd z$.*

Then there exists a z in M such that $z \lhd y$ implies $z=y$.

Proof. For any x in M, we set $S(x)=\{y \in M \mid$ there is a finite number of elements $x_0, x_1, \ldots x_n$ in M such that $x_0=x$, $x_n=y$ and $x_{i-1} \lhd x_i$, $1 \leqq i \leqq n\}$. Then note that $y \in S(x)$ implies that $S(y) \subset S(x)$. Take arbitrary element $x_1 \in M$. Since ϕ is bounded from below, by using induction we have a sequence $\{x_n\}$ in M with $x_{n+1} \in S(x_n)$ and $\phi(x_{n+1}) \leqq \inf\{\phi(y) \mid y \in S(x_n)\} + 1/n$, $n \geqq 1$. Then the condition (5) gives us that there is an element z in M with $z \in S(x_n)$ for each n. Hence by (4) we know that the sequence $\{\phi(x_n)\}$ is nonincreasing and $\phi(z) \leqq \phi(x_n) < \phi(z) + 1/n$ for $n=1, 2, \ldots$. So $\lim \phi(x_n) = \phi(z)$. Now suppose that $z \lhd y$ for some $y \in M$. Then we have $y \in S(x_n)$ for all $n \geqq 1$, so that $\phi(y) \leqq \phi(x_{n+1}) \leqq \phi(y) + 1/n$, $n=1, 2, \ldots$. Therefore we have $\phi(y) = \phi(z)$, and hence by (4) we see that $y=z$.

REMARK 3.2. In Lemma 3.1, the relation $\lhd$ need not be an order relation since it is not assumed to be transitive. However this relation has transitivity implicitly. In fact, if we define another relation $\leq$ on M by $x \leq y$ if and only if $y \in S(x)$, then $\leq$ is an order on M, and we can apply Brézis–Browder's result in [4] to this order to obtain our result.

Proof of Theorem A. Let $w \in B(Tx_0\,;N)$, that is, $d(Tx_0, w) < N$. We can choose a number b with $0<b<\frac{1}{2}$ such that $(1+b)(1-b)^{-1}(1-2b)^{-1}d(Tx_0, w)<N$. Since Y is complete and metrically convex, there is an isometry σ from $[0, d(Tx_0, w)]$ into Y with $\sigma(0)=Tx_0$ and $\sigma(d(Tx_0, w))=w$. Let $M_0=\{x\in D \mid Tx=\sigma(t)$ for some $t\in[0, d(Tx_0, w)]\}$, that is, $M_0=T^{-1}(im\,\sigma)$. Then M_0 is nonempty since $x_0 \in M_0$.

Give a relation $\lhd$ on M_0 such that $x \lhd y$ if and only if $x=y$ or there exists an $\varepsilon>0$ for which

(6) $y\in B(x\,;\varepsilon)\subset D$,

(7) $(1-b)m(x)<m(z)<(1+b)m(x)$ for all $z\in B(x\,;\varepsilon)$,

(8) $(1-2b)\min\{m(x),\ m(y)\}\,d(x, y)\leqq d(Tx, Ty)$ and

(9) $\sigma^{-1}(Tx)\leqq\sigma^{-1}(Ty)$.

Let M be the set of element x in M_0 such that there is a finite number of elements $x_1, x_2, \ldots, x_n$ in M_0 with $x_n=x$ and $x_{i-1}\lhd x_i$ for all $i=1, 2, \ldots, n$, that is $M=S(x_0)$ with the notation as in the proof of Lemma 3.1. Define $\phi: M\to R$ by $\phi(x)=d(Tx_0, w)-\sigma^{-1}(Tx)$. From (9) it can be easily verified that if $x\lhd y$, then $d(Tx, Ty)=\phi(x)-\phi(y)$. Thus we see that the relation on M satisfies the condition (4) by (8). Now we claim that the condition (5) also holds.

Let $\{x_n\}$ be a sequence of elements in M with $x_n\lhd x_{n+1}$ for all $n=1, 2, \ldots$. If necessary, we insert a finite number of elements between x_0 and x_1, we may assume that $x_0\lhd x_1$. Let $\tau_0=0$ and for $n>1$,

$$\tau_n=\textstyle\sum_{i=1}^{n}d(x_{i-1}, x_i) \text{ and } \tau=\sum_{i=1}^{\infty}d(x_{i-1}, x_i).$$

Define a curve $h: [0, \tau)\to D$ such that $h|_{[\tau_{k-1}, \tau_k]}$ $(k\geqq1)$ is the isometry joining from x_{k-1} to x_k. Such an isometry exists, since X is complete and metrically convex, and since $x_{k-1}\lhd x_k$ with the property (6). From its definition we know that h is a continuous curve parametrized by arc length with $h(0)=x_0$. Now we compute that for $k\geqq1$,

$$\begin{aligned}\int_{\tau_{k-1}}^{\tau_k} m(h(t))\,ds_h(t) &\leqq (1+b)m(x_{k-1})(\tau_k-\tau_{k-1})\\ &\leqq\frac{1+b}{1-b}\min\{m(x_{k-1}), m(x_k)\}\,d(x_k, x_{k-1})\\ &\leqq\frac{1+b}{(1-b)(1-2b)}(\phi(x_{k-1})-\phi(x_k)).\end{aligned}$$

Therefore we have

$$\int_0^\tau m(h(t))\,ds_h(t) \leqq \sum_{i=1}^{\infty}\frac{1+b}{(1-b)(1-2b)}(\phi(x_{k-1})-\phi(x_k))$$
$$\leqq \frac{1+b}{(1-b)(1-2b)}d(Tx_0, w)<N,$$

since $\{\phi(x_n)\}$ is a nondecreasing sequence of nonnegative real numbers with $\phi(x_0)=d(Tx_0, w)$. As a result, h is rectifiable and $\lim_{t\to\tau-}h(t)=z\in D$ exists by hypothesis (1). This shows that $x_n\to z$ as $n\to\infty$. Also since $\{\sigma^{-1}(Tx_n)\}$ is nondecreasing and bounded from above, it converges to some $t_0\in[0, d(Tx_0, w)]$. Since σ is an isometry, this shows that $Tx_n\to\sigma(t_0)$. Since T has closed graph, it follows that $Tx_n\to Tz=\sigma(t_0)$. Note that in this case $\phi(x_n)\to\phi(z)$ and $d(Tx_n, Tz)=\phi(x_n)-\phi(z)$ hold.

Now take a number c, $0<c<1$, such that $(1+c)(1-c)^{-1}<\min\{1+b, (1-b)^{-1}\}$. Since m is continuous, there is an $\varepsilon>0$ such that $B(z\,;3\varepsilon/2)\subset D$ and for all $y\in B(z\,;3\varepsilon/2)$,

$$(1-c)m(z)<m(y)<(1+c)m(z).$$

Then note that for each $x\in B(z\,;\varepsilon/2)$, we have $B(x\,;\varepsilon)\subset D$ and (7) holds for each $y\in B(x\,;\varepsilon)$, since

$$(1-b)m(x)<(1-b)(1+c)(1-c)^{-1}m(y)<m(y)$$

and

$$m(y)<(1+c)(1-c)^{-1}m(x)<(1+b)m(x).$$

Now we show that for any positive integer k there is an $n\geqq k$ such that $x_n\lhd z$. Since $x_n\to z$ as $n\to\infty$, we may assume that $x_n\in B(z\,;\varepsilon/2)$ for each $n\geqq k$. We consider two cases.

Case 1. There is an $n\geqq k$ such that $m(x_n)=\inf\{m(x_i)\mid i>n\}$. Then note that for each $i\geqq n$

$$(1-2b)m(x_n)d(x_i, x_{i+1})\leqq d(Tx_i, Tx_{i+1})=\phi(x_i)-\phi(x_{i+1}).$$

For $j>n$, summing up both sides through n to $j-1$, we get

$$(1-2b)m(x_n)d(x_j, x_n)\leqq\phi(x_n)-\phi(x_j).$$

Letting $j\to\infty$, we have

$$(1-2b)m(x_n)d(z, x_n)\leqq\phi(x_n)-\phi(z)=d(Tx_n, Tz),$$

so that $x_n\lhd z$.

Case 2. Suppose that Case 1 does not occur. Then for each $n\geqq k$, there is a $j>n$ so that $m(x_j)<m(x_n)$. Therefore we have $m(z)\leqq m(x_n)$ for each $n\geqq k$ since $m(x_n)$ converges to $m(z)$. Hence by the same method as above we have

$$(1-2b)m(z)d(x_k, z) \leqq \phi(x_k) - \phi(z) = d(Tx_k, Tz),$$

which also gives $x_k \lhd z$. Observing that in any cases z is in M, we know that the condition (5) holds.

By Lemma 3.1, we have an element $z \in M$ such that if $x \in M$ and $z \lhd x$ then $x=z$. Now we claim $Tz=w$, so that we complete the proof. Suppose that $Tz \neq w$. Then $t_0 = \sigma^{-1}(Tz) < d(Tx_0, w)$. Since m is continuous, there is an $\varepsilon > 0$ such that $B(z\,;\varepsilon) \subset D$ and $(1-b)m(z) < m(x) < (1+b)m(z)$ for all $x \in B(z\,;\varepsilon)$. Now we may assume that $(1-b)m(z)\varepsilon < d(Tx_0, w) - t_0$, and also may assume that $T(B(z\,;\varepsilon))$ contains the ball $B(Tz\,;(1-b)m(z)\varepsilon)$ by hypothesis (2). Hence there is an x in $B(z\,;\varepsilon)$ such that $Tx = \sigma(t_0 + (1-2b)m(z)\varepsilon)$. Since $d(Tx, Tz) = (1-2b)m(z)\varepsilon$ and $d(x, z) < \varepsilon$, we have

$$(1-2b)\min\{m(x),\ m(z)\}\, d(x, z) \leqq d(Tx, Tz),$$

which gives gives $z \lhd x$ and $z \neq x$. Since $x \in M_0$ and $z \in M$, we have $x \in M$, which leads a contradiction. Thus $Tz=w$ and the proof is completed.

4. Locally Expansive Operators

Our first concern is to calculate the explicit range of locally expansive operators. Here we consider a local expansion which need not be locally one-to-one and not continuous.

THEOREM 4.1. *Let X and Y be complete metric spaces with both metric convexities, D an open subset of X and let $x_0 \in D$. Let $T: D \to Y$ be an open mapping (that is, T maps open subsets of D onto open subsets of Y) having closed graph, and let $m: D \to (0, \infty)$ be a continuous function satisfying the condition* (1). *Suppose that T is locally expansive in the sense that*

(10) *each point x in D has a neighborhood U in D such that*

$$\min\{m(x), m(u)\}\, d(x, u) \leqq d(Tx, Tu),\ u \in U.$$

Then $T(D)$ contains the ball $B(Tx_0\,;N)$.

Proof. According to Theorem A, it suffices to prove that T satisfies the condition (2). We use a fixed point theorem [2, Theorem 2.1] which is the localized version of the Caristi–Kirk fixed point theorem [6] that is actually equivalent to Ekeland's minimization theorem [8, 9]. Now the following proposition yields our claim.

PROPOSITION 4.2. *Let X and Y be complete metric spaces and $x_0 \in X$. And let $T : B(x_0 ; K) \to Y, K>0$, be an open mapping having closed graph. Further suppose that Y is metrically convex and there exists a constant $m>0$ such that each point $x \in B(x_0 ; K)$ has a neighborhood U in $B(x_0 ; K)$ satisfying*

(11) $$md(x, u) \leqq d(Tx, Tu), \quad u \in U.$$

Then $T(B(x_0 ; K))$ contains the ball $B(Tx_0 ; mK)$.

Proof. Our proof is a slight modification of that of Theorem 3.2 in [3].

Let $w \in B(Tx_0 ; mK)$. Choose an $\varepsilon>0$ so small that $d(w, Tx_0) \leqq m(K-2\varepsilon)$. Introduce a new metric on the set $M=\bar{B}(x_0 ; K-\varepsilon)$ by setting $\rho(x, y)=\max\{d(x, y), m^{-1}d(Tx, Ty)\}$. Then (M, ρ) is complete and $T : (M, \rho) \to Y$ is continuous by Lemma 2.4. Set $\phi(x)=m^{-1}d(w, Tx)$. Then $\phi : (M, \rho) \to R$ is continuous.

Now suppose that $w \notin T(\bar{B}(x_0 ; K-2\varepsilon))$. We now define a mapping $g : M \to M$ as follows; if $x \in M \backslash \bar{B}(x_0 ; K-2\varepsilon)$, set $g(x)=x_0$ $(\neq x)$, note that in this case $\rho(x, y) \geqq d(x, y) > \phi(x_0)-\phi(x)$. And if $x \in \bar{B}(x_0 ; K-2\varepsilon)$, then we choose a neighborhood U of x such that $U \subset B(x ; \varepsilon)$ and (11) holds for this U. By the metric convexity of Y, there is an isometry $\sigma : [0, d(w, Tx)] \to Y$ with $\sigma(0)=Tx$ and $\sigma(d(w, Tx))=w$. Since $T(U)$ is an open set in Y containing Tx and $d(Tx, w) \neq 0$, we can choose a $g(x) \in U$ such that $T(g(x)) \in \sigma((0, d(w, Tx)])$, so that $g(x) \neq x$, $g(x) \in M$ and $d(Tx, w)=d(Tx, T(g(x)))+d(T(g(x)), w)$. Since $d(x, g(x)) \leqq m^{-1}d(Tx, T(g(x)))$ by (11), we get

$$\rho(x, g(x)) \leqq \phi(x)-\phi(g(x)).$$

Hence g has a fixed point in M by Theorem 2.1 of [2], which contradicts the construction of $g(x)$. So $w \in T(B(x_0 ; K-2\varepsilon))$ and the proof is completed.

REMARK 4.3. Note that we do not assume that X is metrically convex in Proposition 4.2. If we put $m(x)= c(d(x_0, x))$ on $D \equiv B(x_0 ; K)$, where $c : [0, K) \to (0, \infty)$ is a continuous function with $\int_0^K c(t)\,dt=N$, then it can be easily proved that the condition (1) holds. In this case, if we follow the same line of the proof of Theorem 3.4 in [3], we can see that $T(B(x_0\,;K))$ contains $B(Tx_0\,;N)$.

Clearly the condition (3) implies (10), but the converse does not hold. Also note that if we adopt the idea of [1] and if the condition (10) is replaced by the condition (3), then the result of Theorem 4.1 is still valid without assuming that X is metrically convex. Combining these results we have the following surjectivity.

Corollary 4.4. *let X and Y be complete metric spaces, D an open subset of X and let $x_0\in D$. Suppose that Y is metrically convex and $m: D\to(0,\infty)$ is a continuous function satisfying the condition*:

(12) *if $h:[0,\tau)\to D$, $0<\tau<\infty$, is a locally rectifiable curve with $h(0)=x_0$ and*

$$\int_0^\tau m(h(t))\,ds_h(t)<\infty,$$

then h is rectifiable and $\lim_{t\to\tau^-}h(t)\in D$.

Let $T: D\to Y$ be an open mapping having closed graph.

(*a*) *If T is locally m-expansive, then T is onto.*

(*b*) *If T satisfies the condition* (11) *for any $x\in D$ and some neighborhood U of x in D, and X is metrically convex, then T is onto.*

Remark 4.5. If $D=X$ in Corollary 4.4, then the condition (12) can be replaced by

(13) if $h:[0,\tau)\to X$ is a locally rectifiable curve with

$$\int_0^\tau m(h(t))\,ds_h(t)<\infty,$$

then h is rectifiable.

5. Differentiable Operators

Let X and Y be Banach spaces and T a mapping from an open subset D of X into Y. We say that T is *Gâteaux differentiable* if for each $x\in D$, there is a mapping $dTx: X\to Y$ satisfying

$$\lim_{t\to0^+}\frac{T(x+th)-T(x)}{t}=dT_x(h),\quad h\in X.$$

If dT_x is a bounded linear operator and the above limit is attained uniformly for all $h\in X$ with $\|h\|<1$, then T is said to be *Frechet* *differentiable*. Note that in the definition of the Gâteaux derivative we only require that the limit is attained on the right hand side, and we do not require that dT_x is linear, but it follows from the definition that $dT_x(th)=tdT_x(h)$ for all $t\geqq 0$. A simple example illustrates this; let $T: R\to R$ be defined by $T(x)=2x$ for $x\geqq 0$ and $T(x)=x$ for $x<0$. Then T is not differentiable at $x=0$, but it is Gâteaux differentiable there. The next theorem can be applied to this mapping. The domain invariance and surjectivity theorems for Gâteaux differentiable operators have been studied by many authors [3, 7, 13, 14, 15].

Now we state our results.

THEOREM 5.1. *Let X and Y be Banach spaces, T a Gâteaux differentiable mapping from an open subset D of X into Y having closed graph and let $x_0\in D$. Suppose that $m: D\to(0,\infty)$ is a continuous function satisfying the condition* (1) *and that, for each $x\in D$,*

$$dT_x(\bar{B}(0\,;\,1))\supset\bar{B}(0\,;\,m(x)).$$

Then $T(D)$ contains the ball $B(Tx_0\,;\,N)$.

Proof. By virtue of Theorem A, it suffices to show that T satisfies the condition (2). But the condition (2) follows from Theorem 3.1 of [3].

REMARK 5.2. As a direct consequence of Theorem 5.1, we can easily obtain the following:

(a) If m satisfies the condition (12), then $T(D)=Y$.

(b) If $D=X$ and m satisfies the condition (13), then $T(X)=Y$.

Let T be a C^1 mapping of X into Y. It is well-known that if dT_x is an isomorphism of X onto Y for all $x\in X$ with $\|[dT_x]^{-1}\|\leqq 1/m(x)$, where $m: X\to(0,\infty)$ is a continuous function. The next lemma will be needed in the sequel.

LEMMA 5.3 *Let X and Y be Banach spaces, D an open subset of X and $T: D\to Y$ a C^1 mapping such that for each $x\in D$, dT_x is an isomorphism of X onto Y with $\|[dT_x]^{-1}\|\leqq 1/m(x)$, where $m: D\to(0,\infty)$ is a continuous function. Then*

(a) for each $x\in D$, $dT_x(\bar{B}(0\,;\,1))\subset\bar{B}(0\,;\,m(x))$ and

(b) for each ε, $0<\varepsilon<1$, the mapping T is locally εm-expansive.

Proof. (a) is clear. By the inverse mapping theorem T is a local diffeomorphism. Therefore there are a neighborhood U of x in D and a neighborhood V of Tx in Y such that T is a C^1 diffeomorphism of U onto V. Since m is continuous we may assume that $\sqrt{\varepsilon}\, m(x) < m(u) < m(x)/\sqrt{\varepsilon}$ for all $u \in U$ and V is convex. For any $u, v \in U$, by putting $x_t = T^{-1}(tTu + (1-t)Tv) \in U, 0 \leqq t \leqq 1$, we have

$$\begin{aligned} \|u-v\| &= \left\| \int_0^1 d(T^{-1})_{tTu+(1-t)Tv}(Tu-Tv)\,dt \right\| \\ &= \left\| \int_0^1 [dT_{x_t}]^{-1} \ (Tu-Tv)\,dt \right\| \\ &\leq \frac{1}{\sqrt{\varepsilon} m(x)} < \|Tu-Tv\|. \end{aligned}$$

Therefore we have

$$\begin{aligned} \min\{\varepsilon m(u), \varepsilon m(v)\} \|u-v\| &\leq \sqrt{\varepsilon}\, m(x) \|u-v\| \\ &\leq \|Tu-Tv\|, \end{aligned}$$

so that T is locally εm-expansive and the proof of (b) is completed.

THEOREM 5.4. *Let X and Y be Banach spaces, D a nonempty open subset of X and let $T: D \to Y$ be a C^1 mapping such that for each. $x \in D$, dT_x is an isomorphism of X onto Y. Suppose that $\|[dT_x]^{-1}\| < 1/m(x)$ for each $x \in D$, where $m: D \to (0, \infty)$ is a continuous function. If m satisfies the condition* (1) *for some $x_0 \in D$, then $T(D)$ contains the ball $B(Tx_0\,; N)$.*

Proof. Direct from Theorem 5.1 and Lemma 5.3 (a).

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